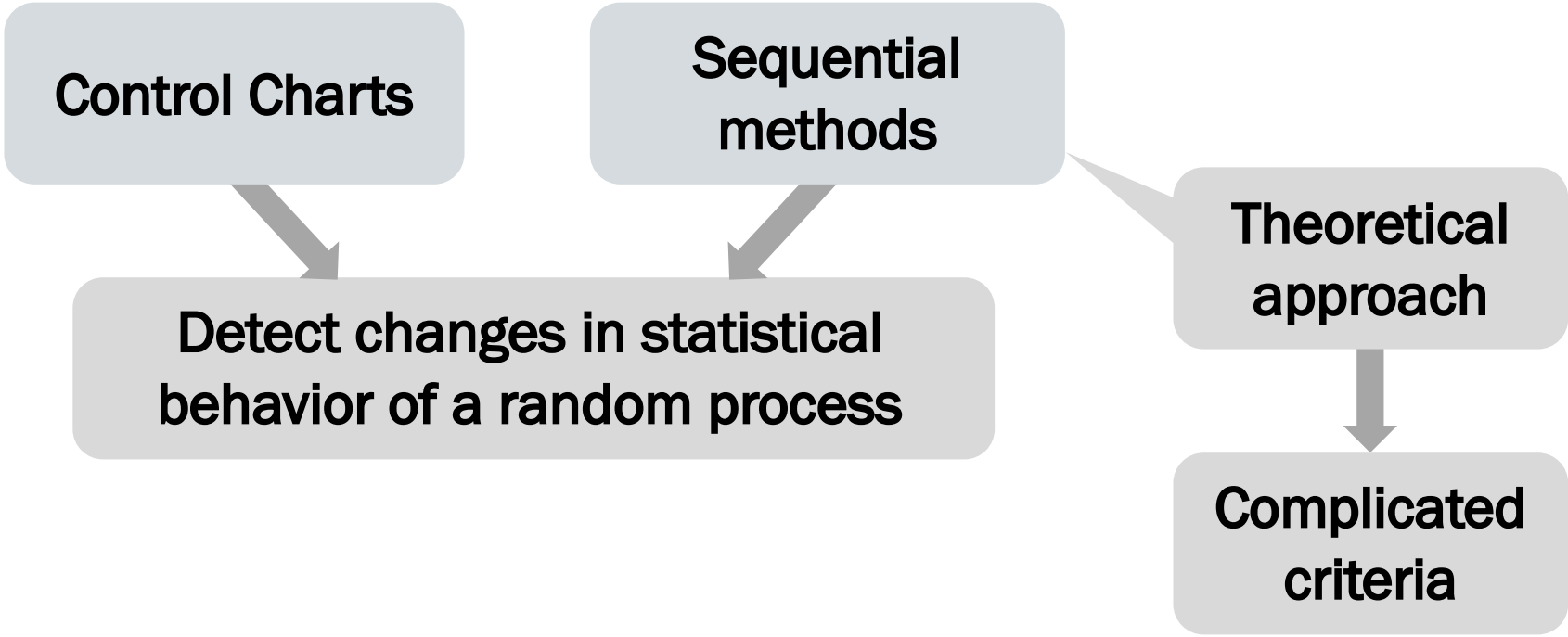
Designing Optimal Tests for Sequential Detection of Changes

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Preview



$$\sup_{t\geq 0} \sup_{X_1,\ldots,X_t} \mathbb{E}_1\left[T-t \,|\, T>t, X_1,\ldots,X_t\right]$$ (1971, Lorden)

Overly pessimistic ?

$$\sup_{t\geq 0} \mathbb{E}_1\left[T-t \,|\, T>t\right]$$ (1985, Pollak)

Outline

Sequential Detection of Changes - Problem Definition

Detection Strategies

Understanding Changes

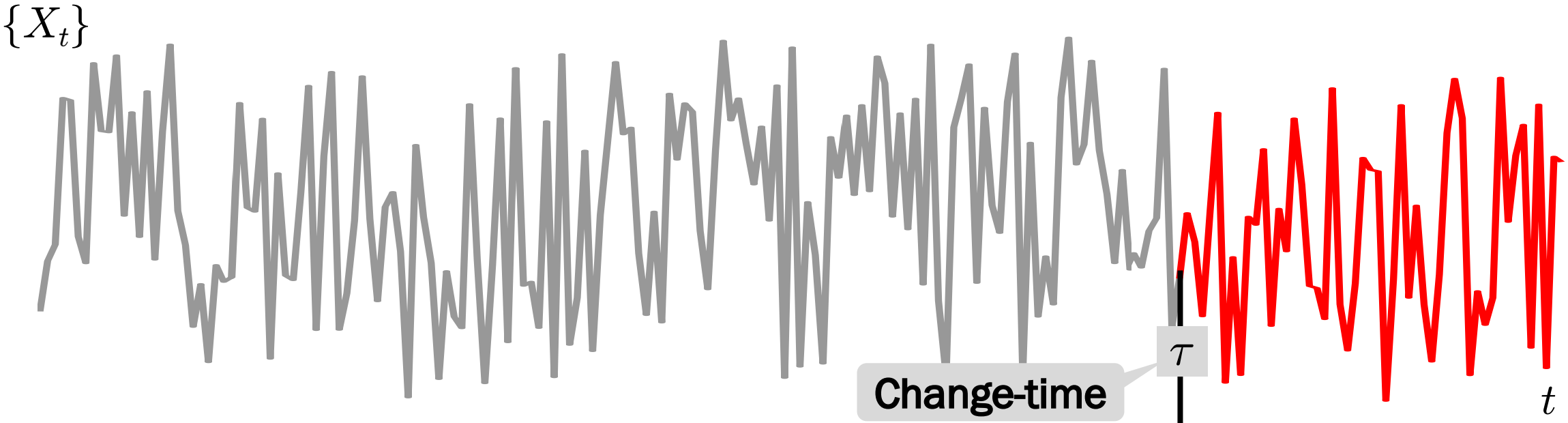
Performance Criteria and Optimal Tests

- Pollak Criterion and the SRP Test
- Lorden Criterion and the CUSUM Test
- Maximal Detection Probability and the Shewhart Test

Take-Home Messages

Problem Definition

Observe sequentially a random process $\{X_t\}$ evolving in time



Control Charts	In Control	Out of Control
Sequential Change Detection	Pre-Change Nominal	Post-Change Alternative

Detect change in statistical behavior **as soon as possible**

Quality Monitoring

Epidemic Detection

Epilepsy Episode Detection

Structural Change-detection in Exchange Rates

Portfolio Monitoring

Electronic Communications

Seismology

Speech & Image Processing (segmentation)

Vibration Monitoring (Structural health monitoring)

Security Monitoring (fraud detection)

Spectrum Monitoring

Scene Monitoring

Network Monitoring (router failures, attack detection)

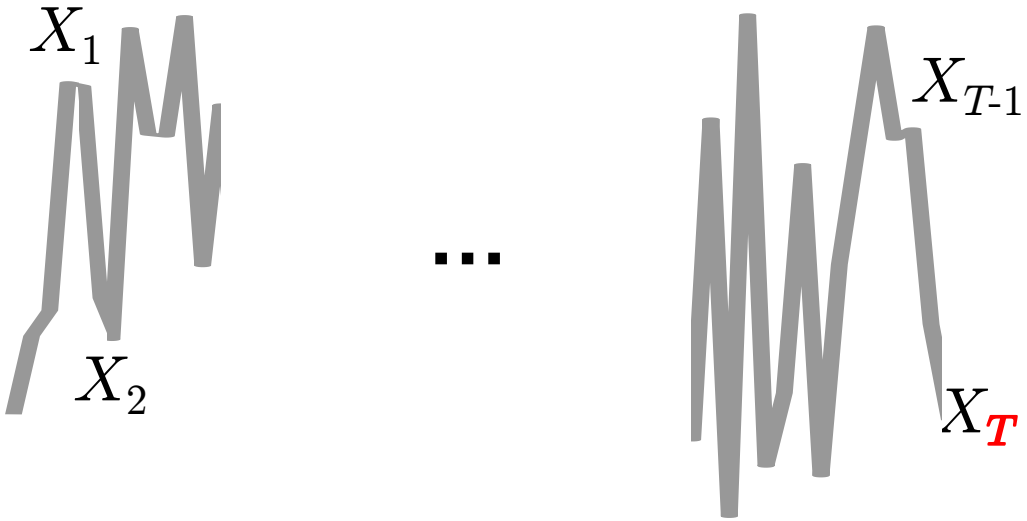
⋮

Detection Strategies

Sequential Test: At each time t use **available** observations $\{X_1,\ldots,X_t\}$

to decide whether a change has occurred at t or before **(no information from future)**

Common tests: Declare change if $S_t(X_1,\ldots,X_t) \geq \nu_t$



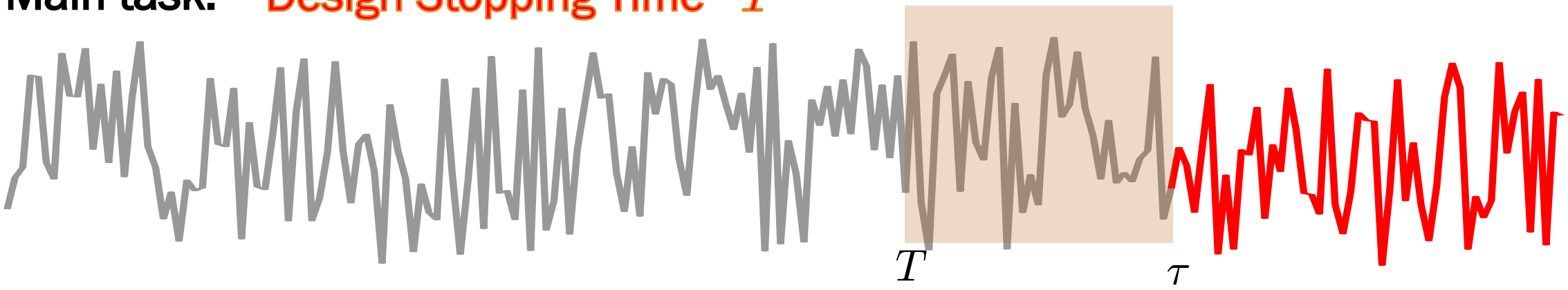
Sequential Test

equivalent to

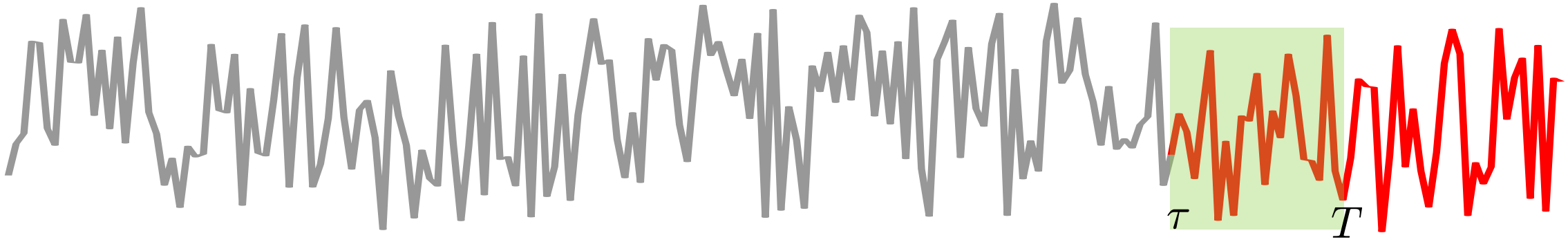
Stopping Time

1	X_1	Was there a change?	No,	take one more sample
2	X_1,X_2	Was there a change?	No,	take one more sample
		⋮		
T-1	$X_1,X_2,\ldots,X_{T\text{-}1}$	Was there a change?	No,	take one more sample
T	$X_1,X_2,\ldots,X_T$	Was there a change?	Yes,	Stop sampling (stopping time)

Main task: Design Stopping Time T

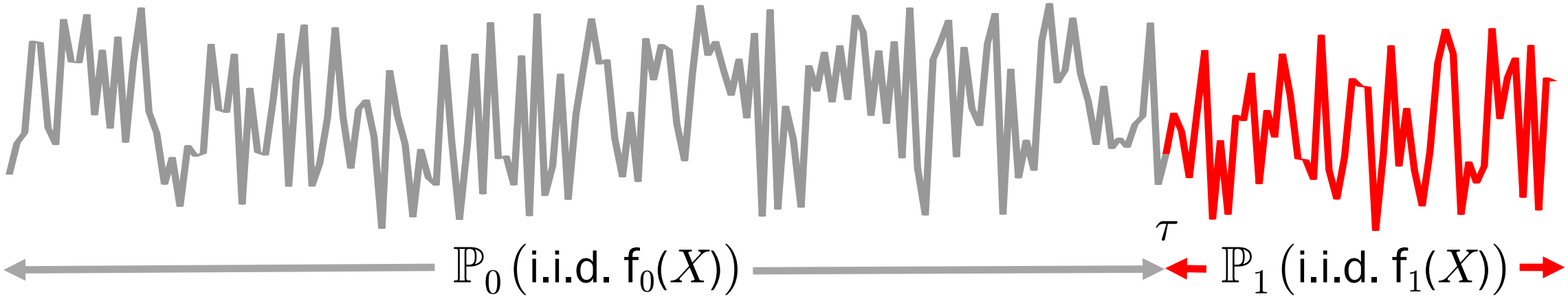


Premature Stopping $\Rightarrow$ False Alarm (infrequent)

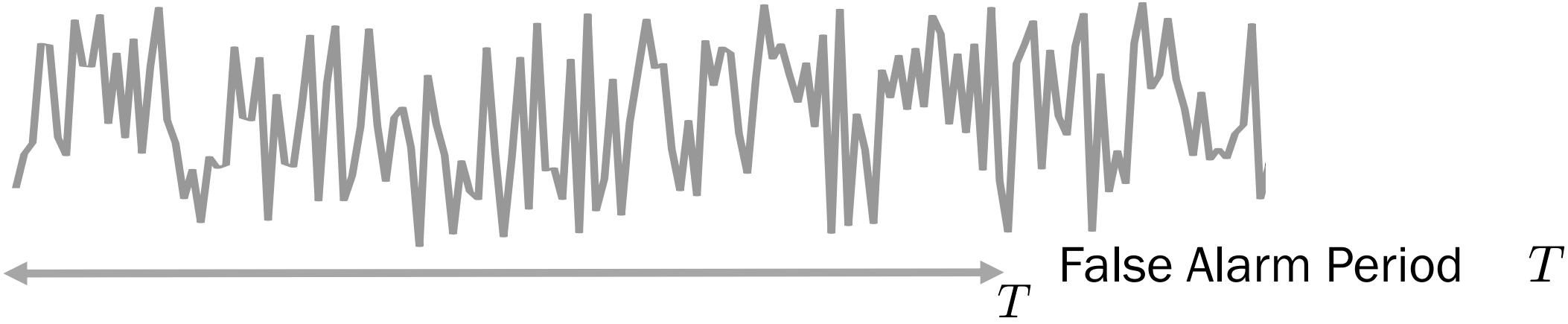


Successful Detection $\Rightarrow$ $T - \tau$ Detection Delay (short)

For the theoretical design of T need to quantify both

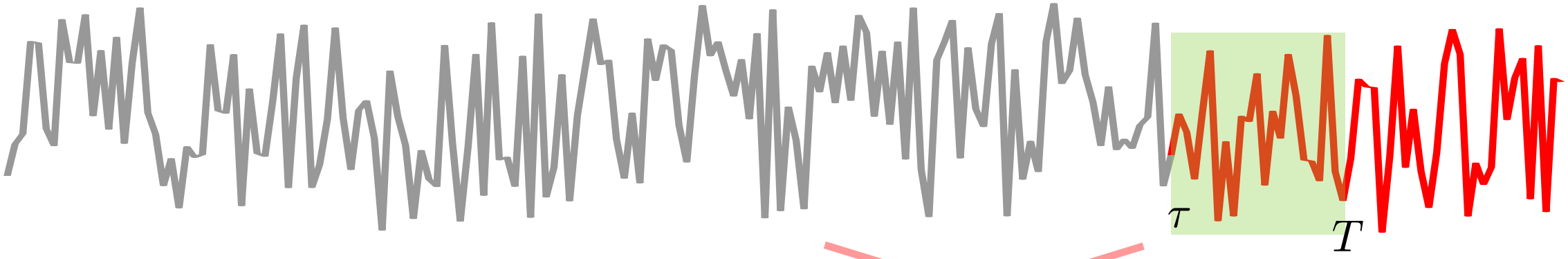


False Alarm



Average False Alarm Period (large): $\mathbb{E}_0[T] \geq \gamma$ **Computable**

Detection Delay



Average Detection Delay: $\mathbb{E}_1[\max\{T-\tau, 0\}]$

Biased due to false alarms

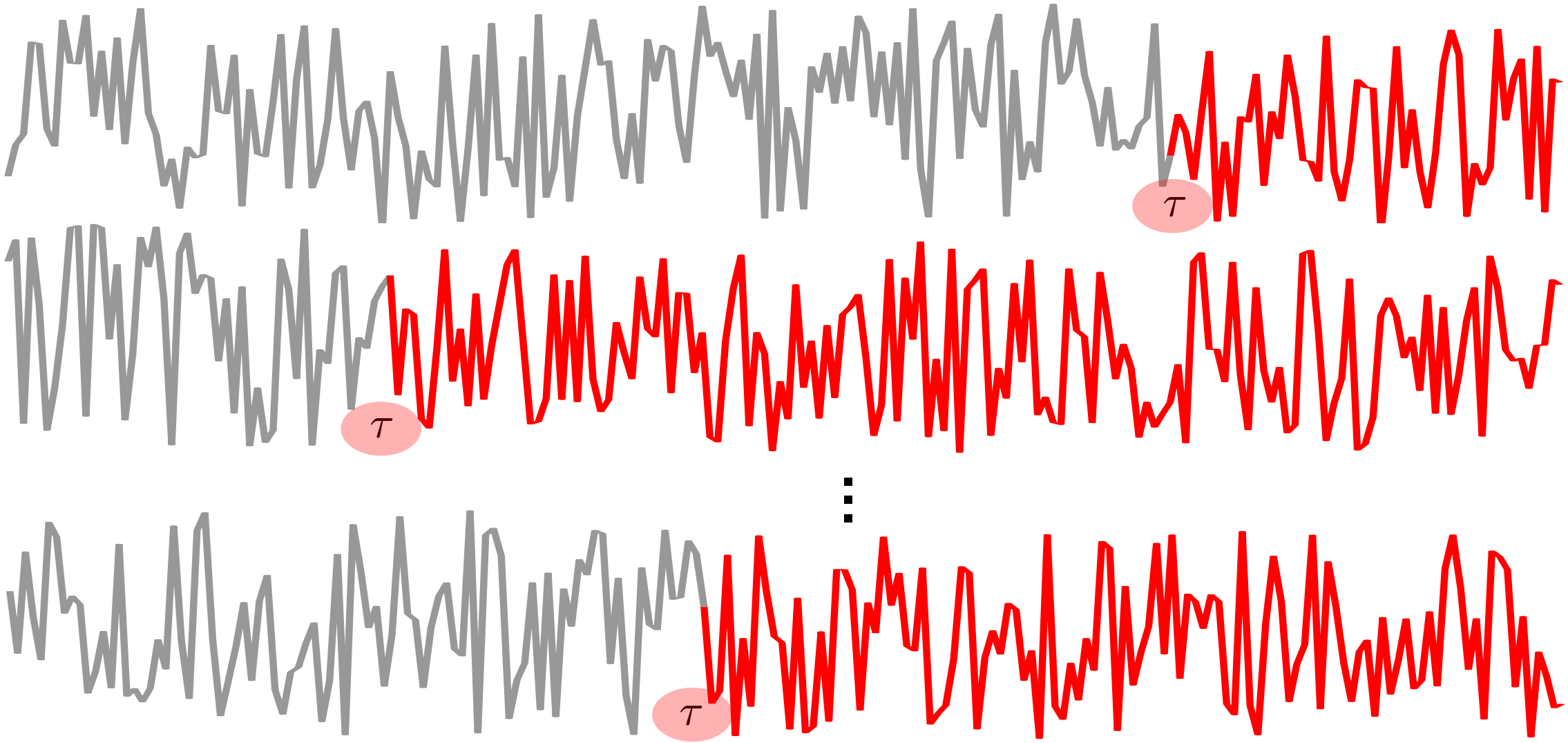
Extreme case: $T = 0$

Conditional Average Detection Delay (small): $\mathbb{E}_1\left[T-\tau \,|\, T>\tau\right]$

Computable ?

What is change-time τ ?

Understanding Changes



Change-time τ random ! Can this randomness be described by $\mathbb{P}_0$, $\mathbb{P}_1$?



Bomb



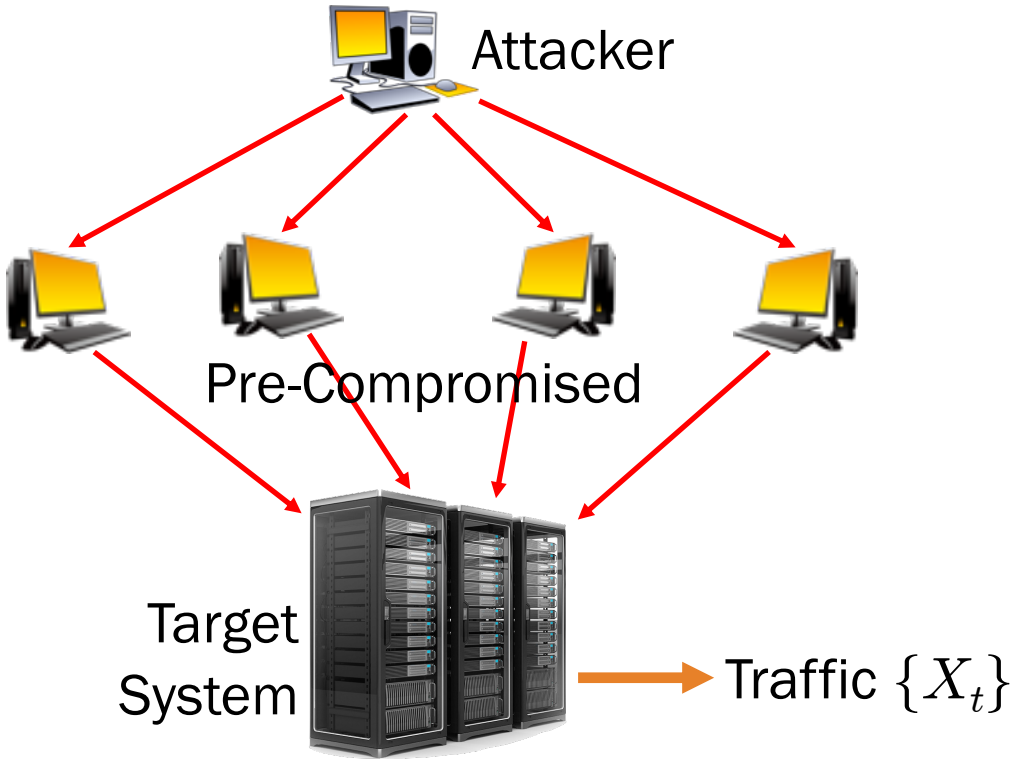


Vibration measurements $\{X_t\}$ at sensors to detect structural changes

Change imposing mechanism does not use observations $\{X_t\}$

Relies on independent data (coordinates of the ball in football game!)

Attack



The system uses traffic measurements $\{X_t\}$ to detect attacks

The attacker has no access to $\{X_t\}$, therefore time of attack is independent from observations



Earthquake

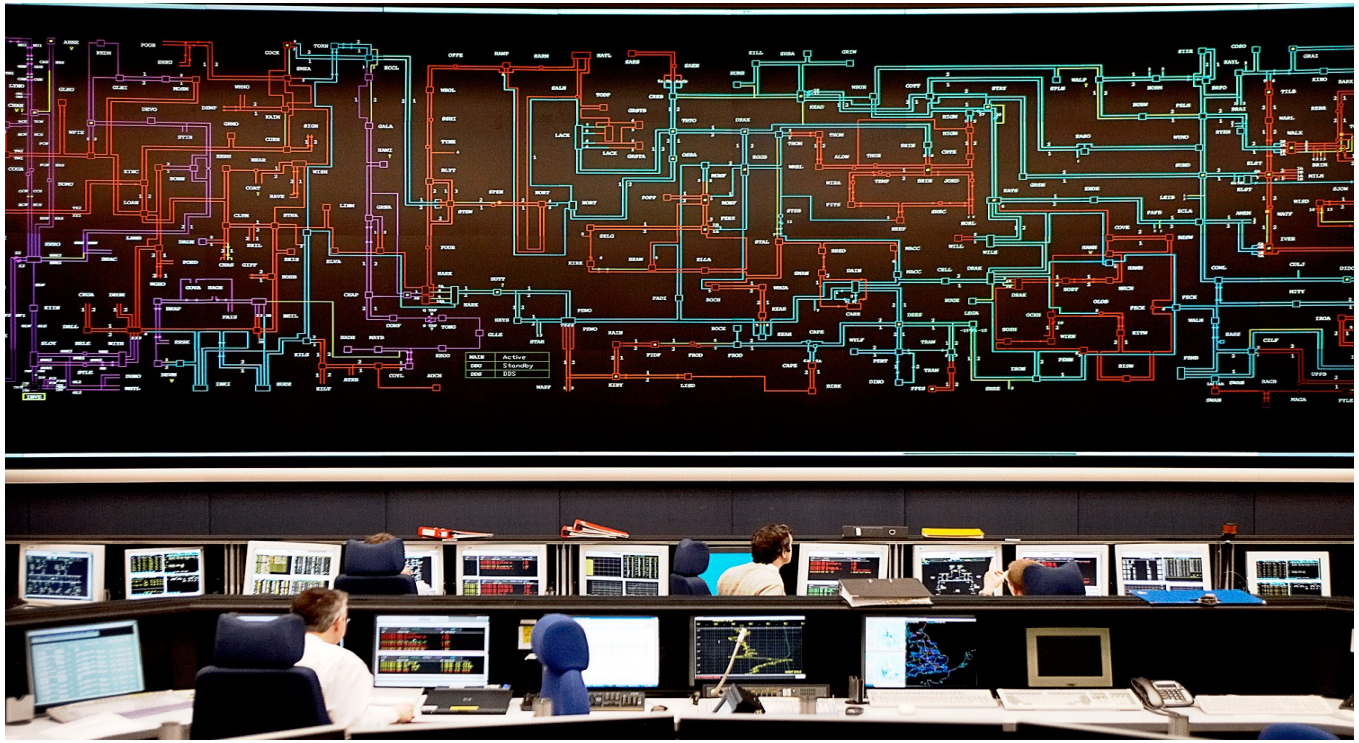
Change imposing mechanism consults $\{Z_t\}$ which depends on observations $\{X_t\}$

Vibration measurements $\{X_t\}$ at sensors to detect structural changes

$X_t = AZ_t + W_t$, Z_t state of the whole structure

Change (crack) when $\|Z_t\|^2 \geq \lambda$

In previous examples detection delay can be arbitrarily large



Power Grid

$\{X_t\}$ measurements at major points

Change in statistical behavior must be detected between τ and $\tau + W$

After $\tau + W$ change produces blackout!

Detection delay is hard limited: $W \geq T - \tau > 0$

$\{X_t\}$ Observations collected sequentially to detect the change

Change Imposing Mechanism

- Mechanism applies changes based on information independent from $\{X_t\}$
- Mechanism applies changes based on information dependent on $\{X_t\}$

Constraint on detection delay

- No hard limit
- Hard limited

Performance Criteria and Optimal Tests

Change Imposing Mechanism Independent from Observations

- No hard limit

$$\mathbb{P}(\tau = t) = \pi_t \quad \text{sequence of numbers}$$

When $\{\pi_t\}$ unknown follow a worst-case analysis

Pollak Criterion

$$J(T) = \sup_{\{\pi_t\}} \mathbb{E}_1\left[T - \tau \,|\, T > \tau\right] = \sup_{t \geq 0} \mathbb{E}_1\left[T - t \,|\, T > t\right]$$

Optimize T by solving constrained optimization (1985)

$$\inf_T J(T) = \inf_T \sup_{t \geq 0} \mathbb{E}_1\left[T - t | T > t\right]$$

subject to: $\mathbb{E}_0\left[T\right] \geq \gamma$

For i.i.d. observations before and after the change
Pollak proposed the Shiryaev-Roberts-Pollak (SRP) Test

At each time t with new sample X_t update statistic S_t

$$S_t = (1 + S_{t-1}) \frac{\mathrm{f}_1(X_t)}{\mathrm{f}_0(X_t)}$$

$$T_{\mathrm{SRP}} = \inf \{ t > 0 : \; S_t \geq \nu \}$$

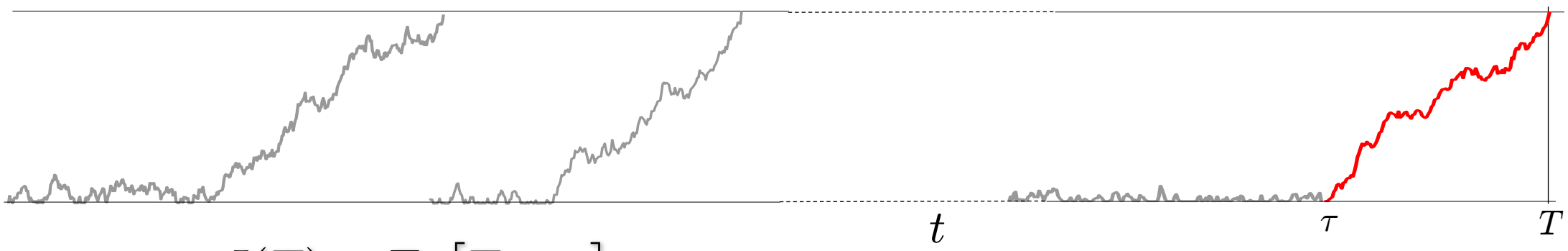
Select ν to satisfy false alarm constraint: $\mathbb{E}_0 [T_{\mathrm{SRP}}] = \gamma$

Specially design initial value S_0

(1985) Asymptotically optimum for γ large (tending to infinity)
Strongest sense of asymptotic optimality

NOT exactly optimum (2010 counterexample)

Variant: Multi-Cyclic Detection



$$J(T) = \mathbb{E}_1\left[T - \tau\right]$$

$$\inf_T J(T) = \inf_T \mathbb{E}_1\left[T - \tau\right]$$

Average false alarm period no less than γ

Many (infinite) number of false alarms before detection

Shiryaev-Roberts Procedure

For i.i.d. before and after change:
With new X_t **update statistic** S_t

$$S_t = \left(1 + S_{t-1}\right)\frac{\mathsf{f}_1(X_t)}{\mathsf{f}_0(X_t)} \qquad S_0 = 0$$

$$T_{\mathsf{SR}} = \inf\left\{t > 0 : S_t \geq \nu\right\}$$

$$\nu: \quad \mathbb{E}_0\left[T_{\mathsf{SR}}\right] = \gamma$$

Change Imposing Mechanism Dependent on Observations

- No hard limit

$$\mathbb{P}(\tau = t) = \pi_t(X_1, X_2, \ldots, X_t)$$

The probabilities depend on the realization of observations

When $\{\pi_t(X_1, X_2, \ldots, X_t)\}$ unknown follow a worst-case analysis

$$J(T) = \sup_{\{\pi_t\}} \mathbb{E}_1\left[T - \tau \,|\, T > \tau\right] = \sup_{t \geq 0} \sup_{X_1, \ldots, X_t} \mathbb{E}_1\left[T - t \,|\, T > t, X_1, \ldots, X_t\right]$$

Lorden Criterion

Optimize T by solving constrained optimization (1971)

$$\inf_T J(T) = \inf_T \sup_{t \geq 0} \sup_{X_1, \ldots, X_t} \mathbb{E}_1\left[T - t | T > t, X_1, \ldots, X_t\right]$$

subject to: $\mathbb{E}_0\left[T\right] \geq \gamma$

For i.i.d. data before and after the change apply CUSUM Test

At each time t with new sample X_t update CUSUM statistic S_t

$$S_t = \max\{S_{t-1}, 0\} + \log\left(\frac{\mathrm{f}_1(X_t)}{\mathrm{f}_0(X_t)}\right), \quad S_0 = 0$$

$$T_{\text{CUSUM}} = \inf\{t > 0 : \; S_t \geq \nu\}$$

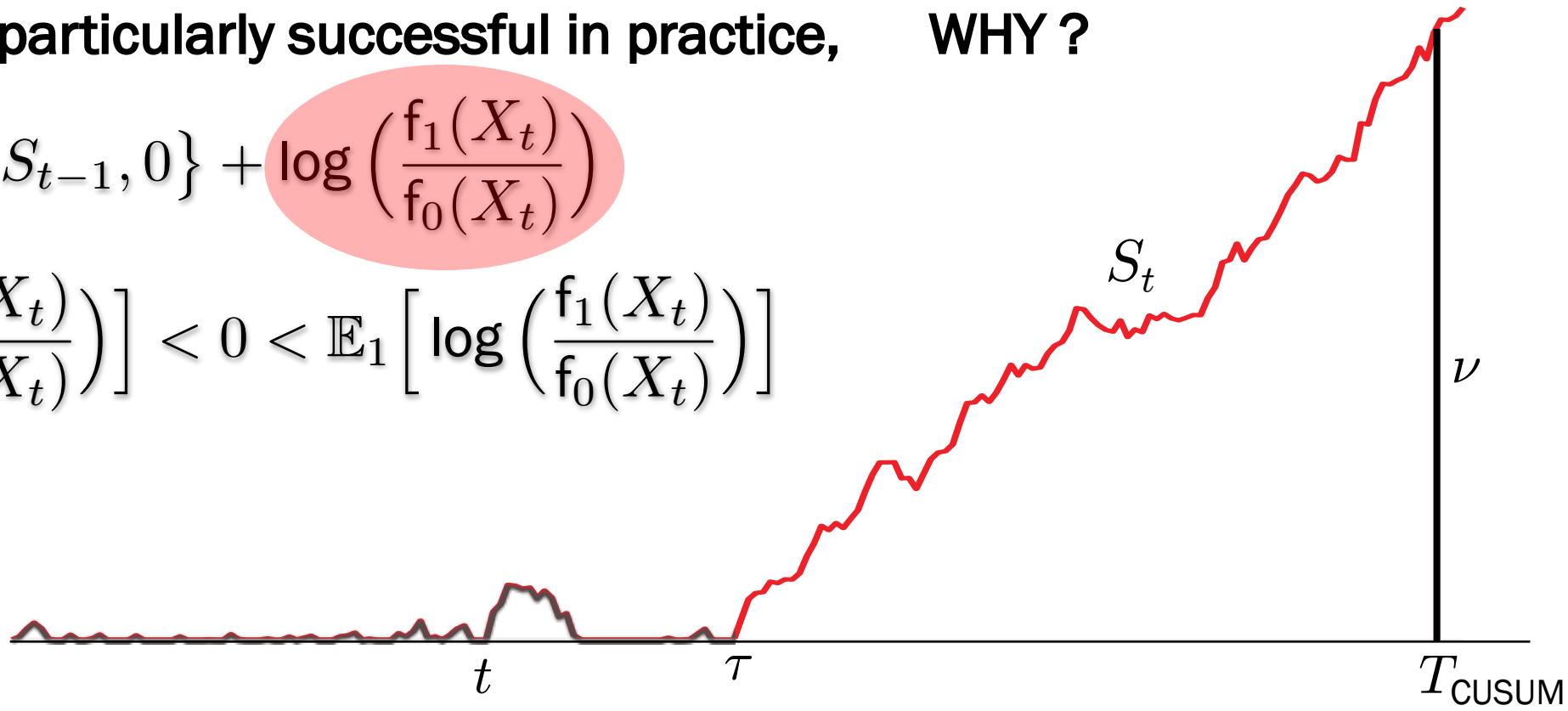
Select ν to safisfy false alarm constraint: $\mathbb{E}_0\left[T_{\text{CUSUM}}\right] = \gamma$

Exact optimality (1986)

CUSUM Test particularly successful in practice, WHY ?

$$S_t = \max\{S_{t-1}, 0\} + \log\left(\frac{\mathrm{f}_1(X_t)}{\mathrm{f}_0(X_t)}\right)$$

$$\mathbb{E}_0\left[\log\left(\frac{\mathrm{f}_1(X_t)}{\mathrm{f}_0(X_t)}\right)\right] < 0 < \mathbb{E}_1\left[\log\left(\frac{\mathrm{f}_1(X_t)}{\mathrm{f}_0(X_t)}\right)\right]$$



Prototype for other data models

$$S_t = \max\{S_{t-1}, 0\} + \log\left(\frac{\mathrm{f}_1(X_t|X_{t-1},\ldots)}{\mathrm{f}_0(X_t|X_{t-1},\ldots)}\right), \quad S_0 = 0$$

No exact optimality

Asymptotic optimality for false alarm values γ large (tending to infinity)

Hard limited detection delay



In certain applications necessary to detected between τ and $\tau + W$
Stopping after $\tau + W$ is no detection (too late)

Interested in T such that $\tau < T \leq \tau + W$

$$\mathbb{P}_1\left(\tau < T \leq \tau + W \,|\, T > \tau\right) = \mathbb{P}_1\left(T \leq \tau + W \,|\, T > \tau\right)$$

Solution only for $W=1$

Immediate detection with the first post-change sample

$$\sup_{T} \inf_{t\geq 0} \mathbb{P}_1\left(T=t+1|T>t\right)$$

or

$$\sup_{T} \inf_{t\geq 0} \inf_{X_1,\ldots,X_t} \mathbb{P}_1\left(T=t+1|T>t, X_1,\ldots,X_t\right)$$

subject to: $\mathbb{E}_0\left[T\right]\geq\gamma$

For i.i.d. data before and after the change optimum is the Shewhart Test

$$T_{\mathsf{sh}}=\inf\left\{t>0:\ \frac{\mathsf{f}_1(X_t)}{\mathsf{f}_0(X_t)}\geq\nu\right\}$$

Take-Home Messages

Changes are imposed by mechanisms

Change mechanism independent from observations $\{X_t\}$

Use Shiryaev-Roberts (Pollak)

Change mechanism depends on observations $\{X_t\}$

Use CUSUM

If unsure

Use CUSUM

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The 25th annual conference of the European Network for Business and Industrial Statistics (ENBIS) will be hosted by the University of Piraeus and will take place at the central building in the city center of Piraeus (Greece), from September 14 to 18. The conference sessions are scheduled from 15th to 17th September, with the administrative meetings, the pre- and post-conference courses and workshops taking place on the 14th and 18th September.



The annual conference features invited and contributed sessions, workshops and panel discussions, pre- and post-conference courses, as well as talks from distinguished keynote speakers and award winners.

This year's keynote speakers include Jennifer van Mullekom (Virginia Tech, USA) and George Moustakides (University of Patras, Greece).

We cordially invite you not only to engage in highly rewarding scientific and professional exchange during the conference, but also to find some leisure time and explore the city of Piraeus, Athens and Greece.