Optimum
Joint Detection & Estimation
Application to MIMO Radar

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Outline

- Joint Detection and Estimation: problem formulation
- Optimum one- and two-step strategies
- MIMO radar
- Application of the two-step scheme
- Simulations

Joint Detection and Estimation

The problem: For a finite sequence of samples $X=[x_1,\ldots,x_n]$ we assume the following two hypotheses:

$$\mathrm{H}_0 : X \sim f_0(X)$$

$$\mathrm{H}_1 : X \sim f_1(X|\theta)$$

Detection: Given the data vector X, **decide** between the two hypotheses

Estimation: Every time there is a decision in favor of H_1 **estimate** θ

Radar





If a target enters the operational space of our radar, we would like to **detect** it.

Once a target is detected we would also like to **estimate** its position, speed…

Detection problem (Neyman-Pearson)

We are given a data vector $X=[x_1,\ldots,x_n]$ for which we assume the following two hypotheses:

$$\mathrm{H}_0: \; X \sim f_0(X)$$

$$\mathrm{H}_1: \; X \sim f_1(X)$$

Use X to select $\mathrm{D} = \mathrm{H}_0$ or H_1

False alarm $\mathrm{P}_0(\mathrm{D}=\mathrm{H}_1) \leqslant \alpha, \; 0<\alpha<1$

Maximize $\mathrm{P}_1(\mathrm{D}=\mathrm{H}_1)$

Minimize $\mathrm{P}_1(\mathrm{D}=\mathrm{H}_0)$

$$\frac{f_1(X)}{f_0(X)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} t$$

$$\mathsf{H}_0 : \; X \sim f_0(X)$$
$$\mathsf{H}_1 : \; X \sim f_1(X|\theta), \quad \pi(\theta)$$

False alarm $\mathsf{P}_0(\mathsf{D} = \mathsf{H}_1) \leqslant \alpha, \; 0 < \alpha < 1$

Minimize $\mathsf{P}_1(\mathsf{D} = \mathsf{H}_0)$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\lessgtr}} t$$

$$\mathsf{P}_1(\mathsf{D} = \mathsf{H}_0) \geqslant \min \mathsf{P}_1(\mathsf{D} = \mathsf{H}_0) = \beta_{\mathrm{NP}}(\alpha)$$

Estimation problem (Bayesian approach)

We are given a data vector $X=[x_1,\ldots,x_n]$ for which we assume the following model:

$$X \sim f_1(X|\theta), \quad \pi(\theta)$$

Use X to find an estimate $\hat{\theta}$ of θ

Minimize $\mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2\right]$

$$\hat{\theta}_o(X) = \mathsf{E}_1[\theta|X] = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

Formulation of the joint problem

$$\mathrm{H}_0: \; X \sim f_0(X)$$

$$\mathrm{H}_1: \; X \sim f_1(X|\theta), \quad \pi(\theta)$$

Ad-hoc

Optimum in each subproblem:

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} t$$

$$\hat{\theta}_o = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

GLRT:

$$\max_{\theta} \frac{f_1(X|\theta)}{f_0(X)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} t$$

One-step tests

We distinguish our decisions into H_0 and H_1 **BUT**

- H_1 : detection and reliable estimation
- H_0 : no detection or detection without reliable estimation

Minimize $\mathsf{E}_1\left[\|\hat{\theta}-\theta\|^2|\mathsf{D}=\mathsf{H}_1\right]$

We need to **control** the detection part

$$\alpha \geqslant \mathsf{P}_0(\mathsf{D}=\mathsf{H}_1) \qquad \beta \geqslant \mathsf{P}_1(\mathsf{D}=\mathsf{H}_0) \geqslant \beta_{\mathrm{NP}}(\alpha)$$

Theorem: The optimum combined detection and estimation scheme is defined as follows:

$$\hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) = \mathsf{E}_1\left[\|\theta - \hat{\theta}_o\|^2 | X\right]$$

$$= \frac{\int \|\theta - \hat{\theta}_o\|^2 f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \left\{\lambda - \sigma^2(X)\right\} \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\lessgtr}} t$$

Segmentation problems



Detect if after some point, there is a change in the statistical behavior of the data.

Every time we detect a change, we like to **estimate** the point of change, i.e. the **boundary**.

Detect objects in images and find boundaries. Detection has no meaning without boundary estimation.

Two-step tests

We distinguish our decisions into H_0 and H_1

Every time I decide in favor of H_1 I compute an estimate for my parameters

I can now ask myself: Can I trust my estimate?

Is the estimate reliable (denoted as H_{1r}) or unreliable (denoted as H_{1u}) ?

Need a **second decision mechanism** that decides between H_{1r} and H_{1u}

We have three decisions: H_0, H_{1r} and H_{1u} (H_1)

Decide H_0 / H_1

IF H_1

Estimate θ

$$\alpha \geqslant P_0(D{=}H_1) \quad \min P_1(D{=}H_0)$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\min E_1\left[\|\hat{\theta} - \theta\|^2 | D = H_{1r}\right]$$

$$\frac{1-\beta}{1-\beta_{NP}(\alpha)} \leq \frac{P(D = H_{1r})}{P(D = H_1)}$$

$$\hat{\theta}_o(X) = \frac{\int \theta f_1(X|\theta)\pi(\theta)d\theta}{\int f_1(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) \underset{H_{1u}}{\overset{H_{1r}}{\lessgtr}} \lambda$$

Decide H_{1r} / H_{1u}

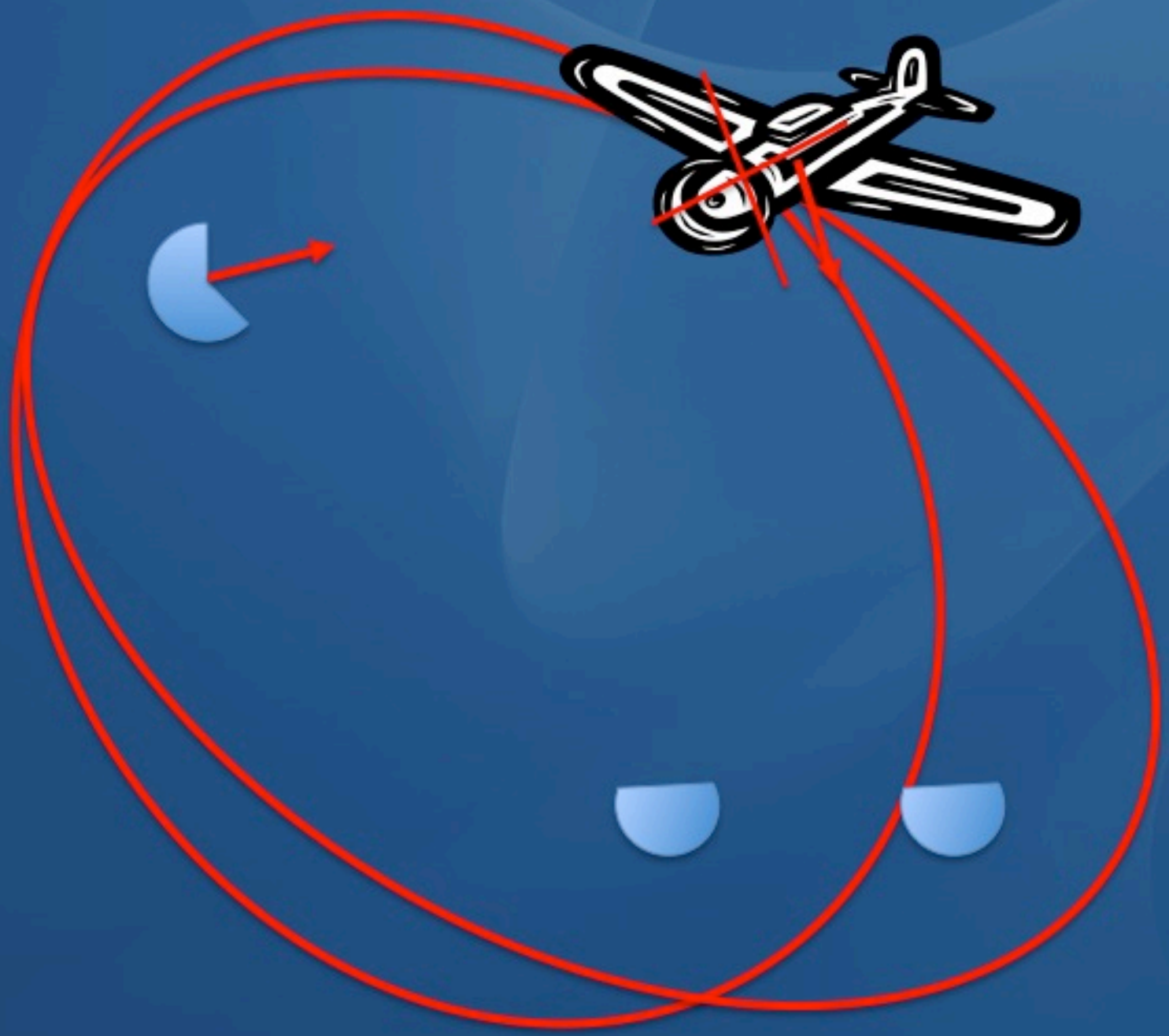
$$\mathcal{L}(X|\theta) = \frac{f_1(X|\theta)}{f_0(X)}$$

$$\frac{\int f_1(X|\theta)\pi(\theta)d\theta}{f_0(X)} = \int \mathcal{L}(X|\theta)\pi(\theta)d\theta \underset{\mathsf{H}_0}{\overset{\mathsf{H}_1}{\gtrless}} t$$

$$\hat{\theta}_o(X) = \frac{\int \theta\mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

$$\sigma^2(X) = \frac{\int \|\theta - \hat{\theta}_o\|^2\mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

MIMO radar



Measure the delay.

The delay is proportional to the distance traveled by the EM wave.

Defines an ellipse.

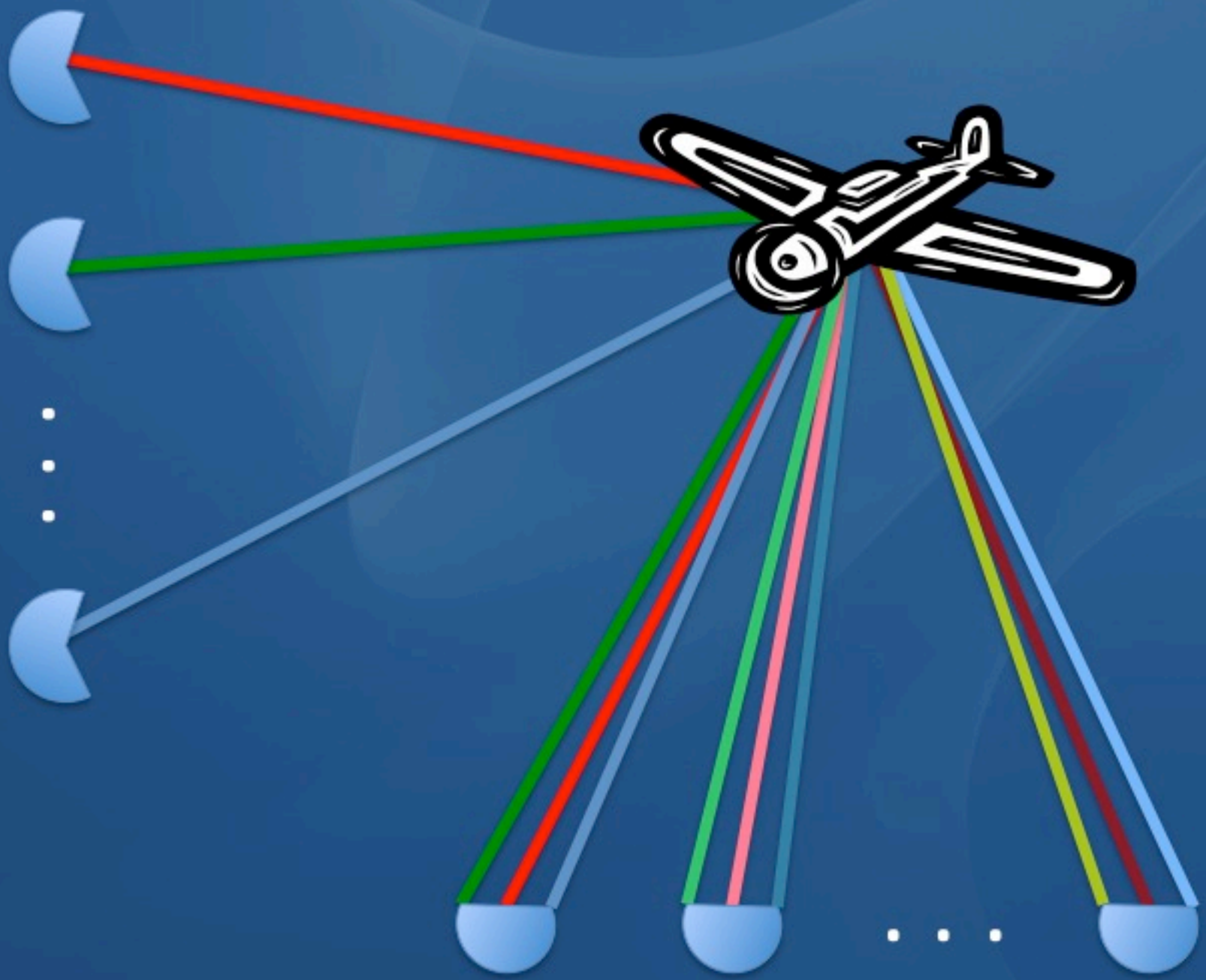
Consider a maximal allowable delay.

Only detect

The operational space of the radar

Detect & estimate

Application of optimum scheme



$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{d_{ij}^{m}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$

$s_1(t)$

$s_2(t)$

$\vdots$

$s_K(t)$

Apply a LR test

$r_1(t)$ $r_2(t)$ $\cdots$ $r_M(t)$

$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$s_1(t)$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$

$s_2(t)$

Treat the delays τ_{ij} as unrelated and estimate them

$\vdots$

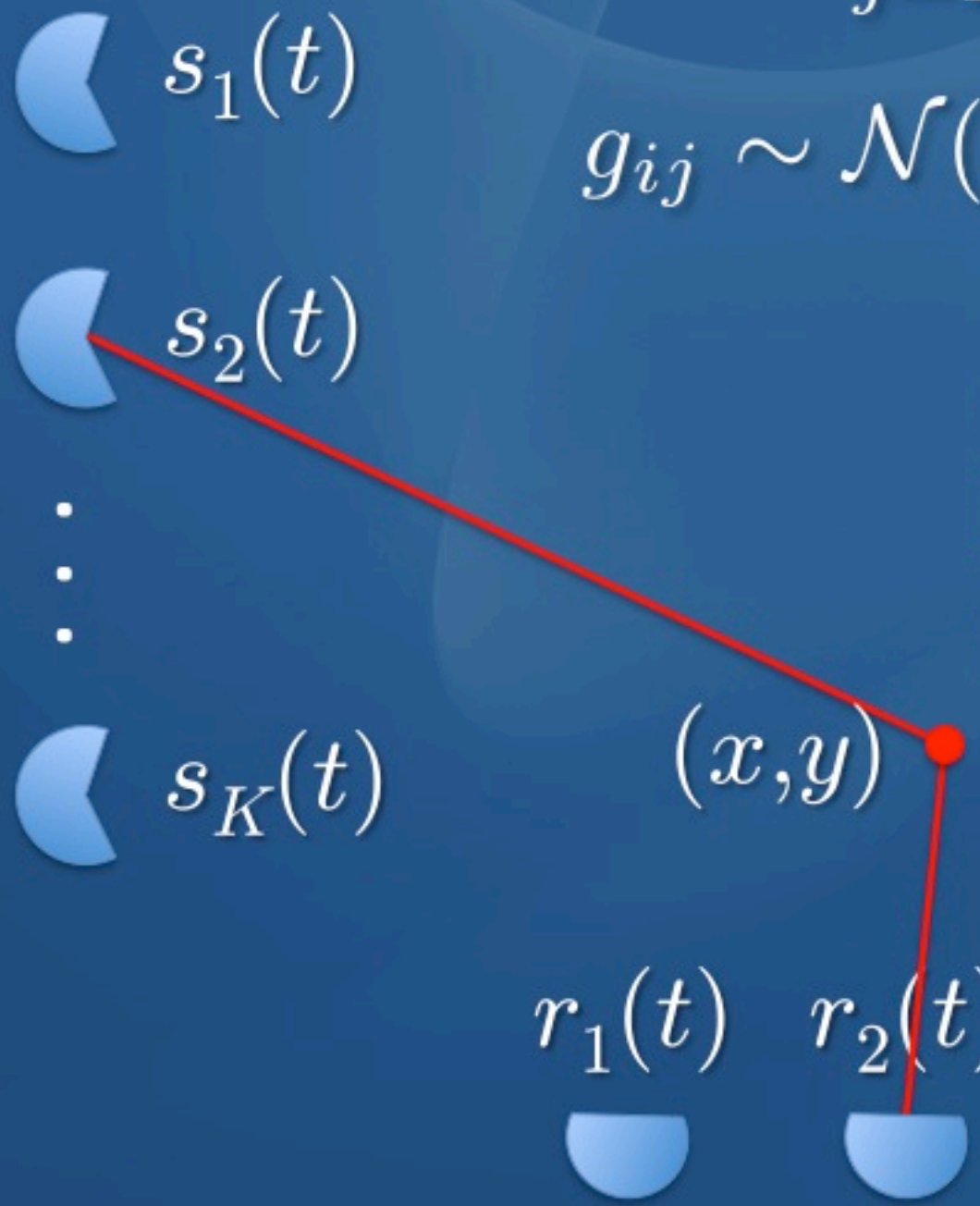
$s_K(t)$

The KN delays generate KN ellipses and a set of at most KN choose 2 target positions. We average.

$r_1(t)$ $r_2(t)$ $\cdots$ $r_M(t)$

$$r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t - \tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$



Actually $\tau_{ij} = \tau_{ij}(x,y)$ are **known** functions of the target position

$$\mathsf{H}_0 : r_i(t) = w_i(t)$$

$$\mathsf{H}_1 : r_i(t) = \sum_{j=1}^{K} g_{ij} \frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}} + w_i(t)$$

$$g_{ij} \sim \mathcal{N}(0,1); \quad w_i(t) \sim \mathcal{N}(0,1)$$

For the position (x,y) we assume **uniform** prior over the operational space of the radar.

$$X \Longleftrightarrow [r_1(t), \ldots, r_M(t)] \qquad \theta \Longleftrightarrow (x,y)$$

$$\mathcal{L}(X|\theta) \Longleftrightarrow \mathcal{L}(r_1(t), \ldots, r_M(t)|x,y)$$

$$\mathcal{L}(r_1(t),\ldots,r_M(t)|x,y)=\prod_{i=1}^{M}\mathcal{L}(r_i(t)|x,y)$$

$$r_i(t)=\sum_{j=1}^{K}g_{ij}\frac{s_j(t-\tau_{ij})}{\tau_{ij}^{\eta}}+w_i(t)$$

$$=G_i'\,S(t-\mathcal{T}_i)+w_i(t)$$

$$G_i=[g_{i1},\ldots,g_{iK}]',$$

iid Gaussian $\mathcal{N}(0,1)$

$$S(t-\mathcal{T}_i)=\left[\frac{s_1(t-\tau_{i1})}{\tau_{i1}^{\eta}},\ldots,\frac{s_K(t-\tau_{iK})}{\tau_{iK}^{\eta}}\right]'$$

Known deterministic signals

Known deterministic functions of (x,y)

Assume we measure $r_i(t)$ during the interval $[0,T]$ then

$$\mathcal{L}(r_i(t)|x,y,G_i) = e^{-0.5G_i'\mathbf{Q}_i(x,y)G_i+G_i'R_i(x,y)}$$

$$\mathbf{Q}_i(x,y) = \int_0^T S(t-\mathcal{T}_i)S'(t-\mathcal{T}_i)dt$$

$$R_i(x,y) = \int_0^T r_i^*(t)S(t-\mathcal{T}_i)dt$$

Integrating out G_i

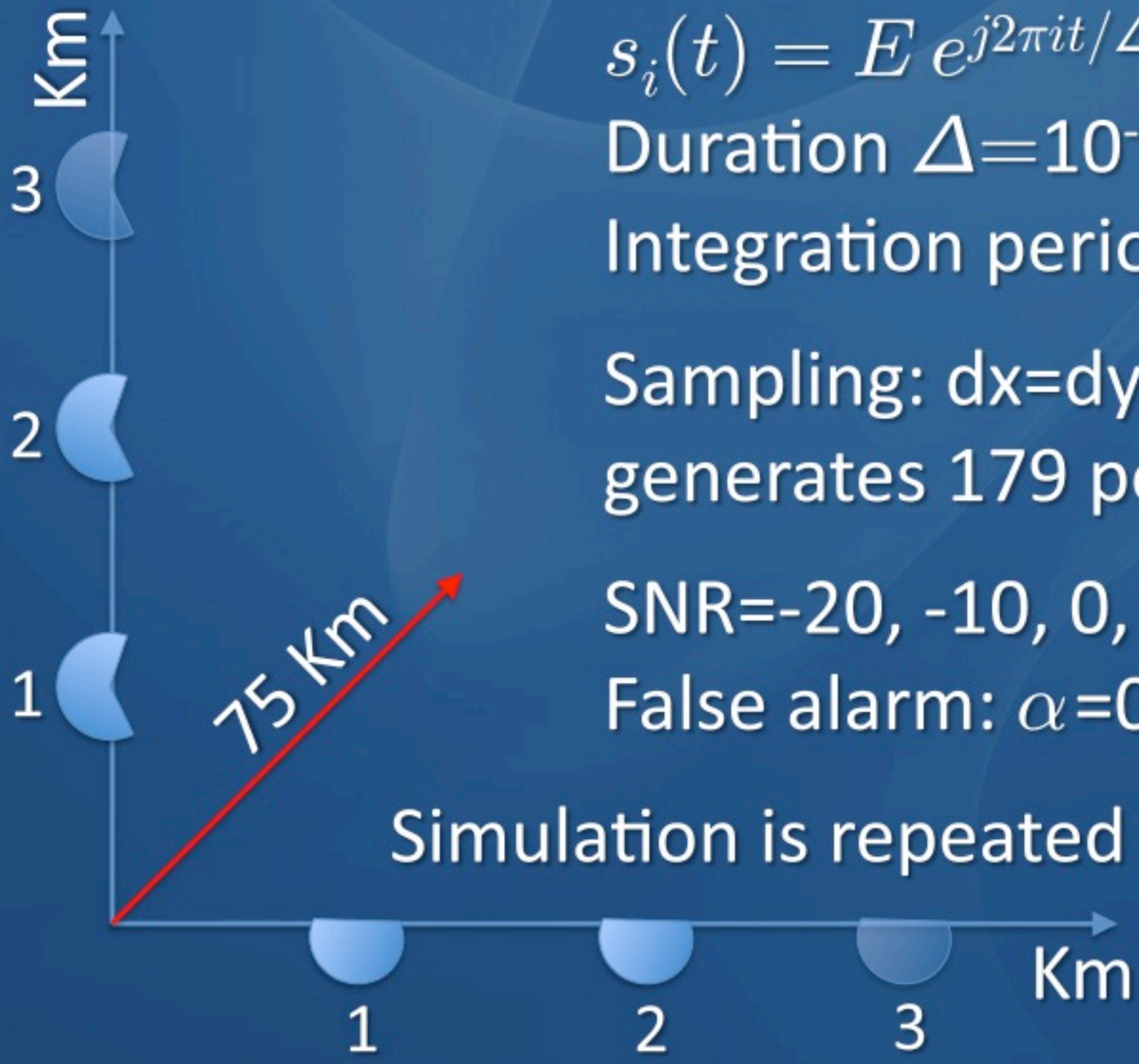
$$\mathcal{L}(r_i(t)|x,y) = \frac{e^{0.5R_i'(x,y)\{\mathbf{Q}_i(x,y)+\mathbf{I}\}^{-1}R_i(x,y)}}{\det(\mathbf{Q}_i(x,y)+\mathbf{I})}$$

$$\hat{\theta}_o = \frac{\int \theta \mathcal{L}(X|\theta)\pi(\theta)d\theta}{\int \mathcal{L}(X|\theta)\pi(\theta)d\theta}$$

Assuming **uniform** prior for the target position on the operational space of the radar and by sampling uniformly this space we have

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \approx \frac{\sum_{(x_k,y_k)\in \text{op.sp.}} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}{\sum_{(x_k,y_k)\in \text{op.sp.}} \prod_{i=1}^{M} \mathcal{L}(r_i|x_k,y_k)}$$

Simulations



$s_i(t) = E\, e^{j2\pi it/\Delta}$

Duration Δ=10^{-4}sec

Integration period T=5x10^{-4}sec

Sampling: dx=dy=10 Km

generates 179 points

SNR=-20, -10, 0, 10 dB

False alarm: α=0.001

Simulation is repeated 200,000 times

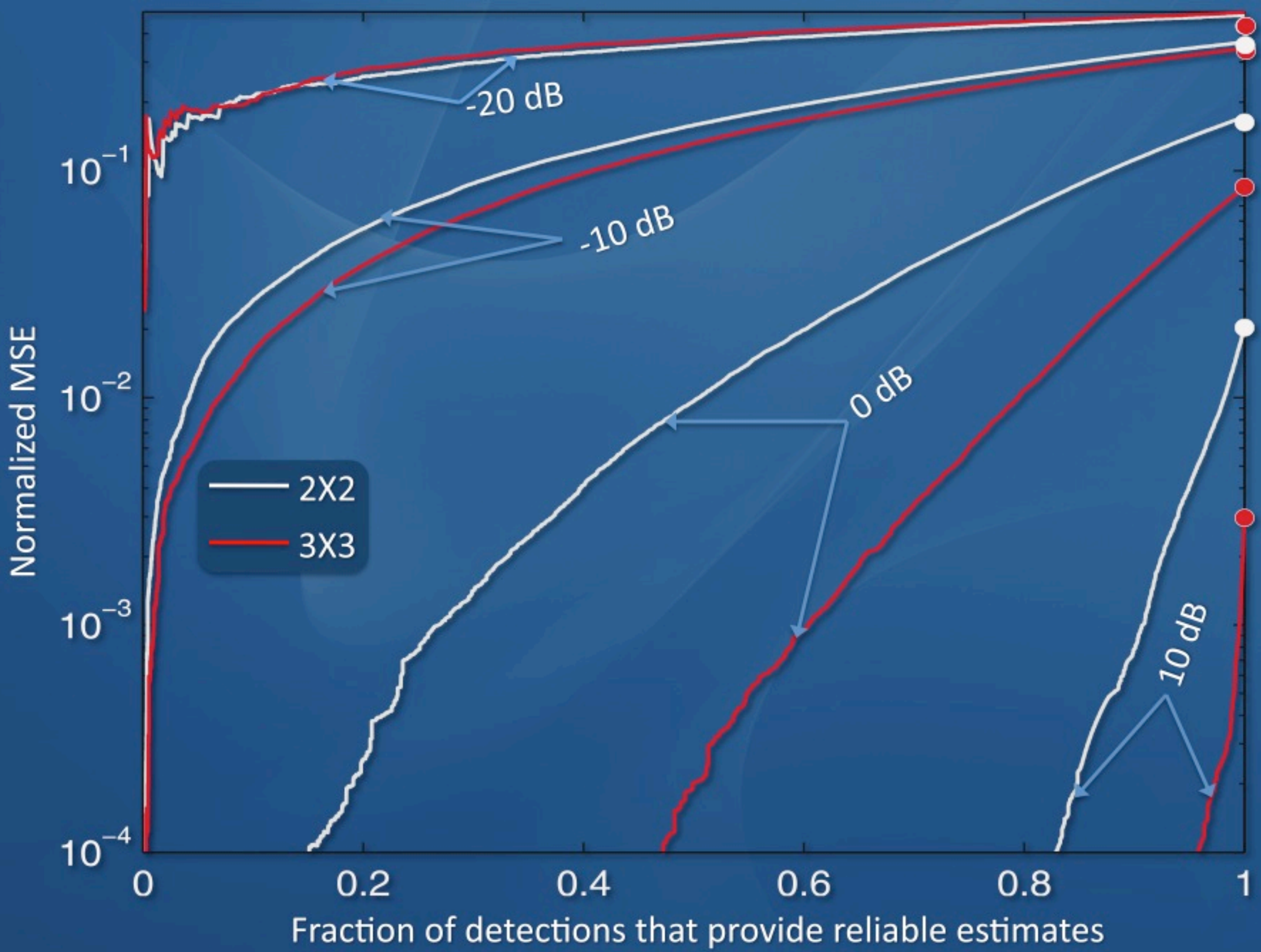
1) GLRT: $$\max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i|x,y) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} t$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \arg\max_{x,y} \prod_{i=1}^{M} \mathcal{L}(r_i|x,y)$$

2) For each subproblem use the optimum.

3) Proposed optimum two-step test.

We note that 2) is a special case of 3)


-20 dB
-10 dB
0 dB
10 dB
2X2
3X3
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Normalized MSE
0
0.2
0.4
0.6
0.8
1
Fraction of detections that provide reliable estimates

USC Viterbi
School of Engineering

Ming Hsieh
Department of Electrical Engineering

Seminar Announcement

Communication Sciences

Optimum Joint Detection and Estimation: Application to MIMO Radar

George Moustakides
University of Patras

Tuesday, January 25, 2011
1:00 pm - 2:00 pm
EEB 248

Abstract: We consider the joint detection and parameter estimation problem. By combining the Baysian formulation for estimation with suitable constraints on the detection subproblem we develop optimum one- and two-step test for the joint detection/estimation case. The proposed combined strategies have the very desirable characteristic to allow for the trade-off between detection power and estimation efficiency. Our theoretical developments are then applied to the problem of MIMO radar where we detect a target but also estimate its location. Simulations demonstrate that by using the jointly optimum schemes, we can experience significant improvement in estimation quality with small sacrifice in detection power.

Biography: George V. Moustakides received the diploma in Electrical & Mechanical Engineering from the National Technical University of Athens, Greece, in 1979; the MSc in Systems Engineering from the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, in 1980, and the PhD in Electrical Engineering and Computer Science from Princeton University, Princeton NJ, in 1983. From 1983 to 1986 he was with INRIA, France and from 1987 to 1990 with the Computer Technology Institute of Patras, Greece. In 1991 he joined the Computer Engineering and Informatics department, University of Patras, Greece as Associate Professor and became Professor in 1996. Since 2007 he is with the department of Electrical and Computer Engineering of the University of Patras. During the period 2001-2004 he also held a senior researcher position with INRIA, France.

His current interests include: sequential detection, statistical signal processing and signal processing for hearing aids.

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