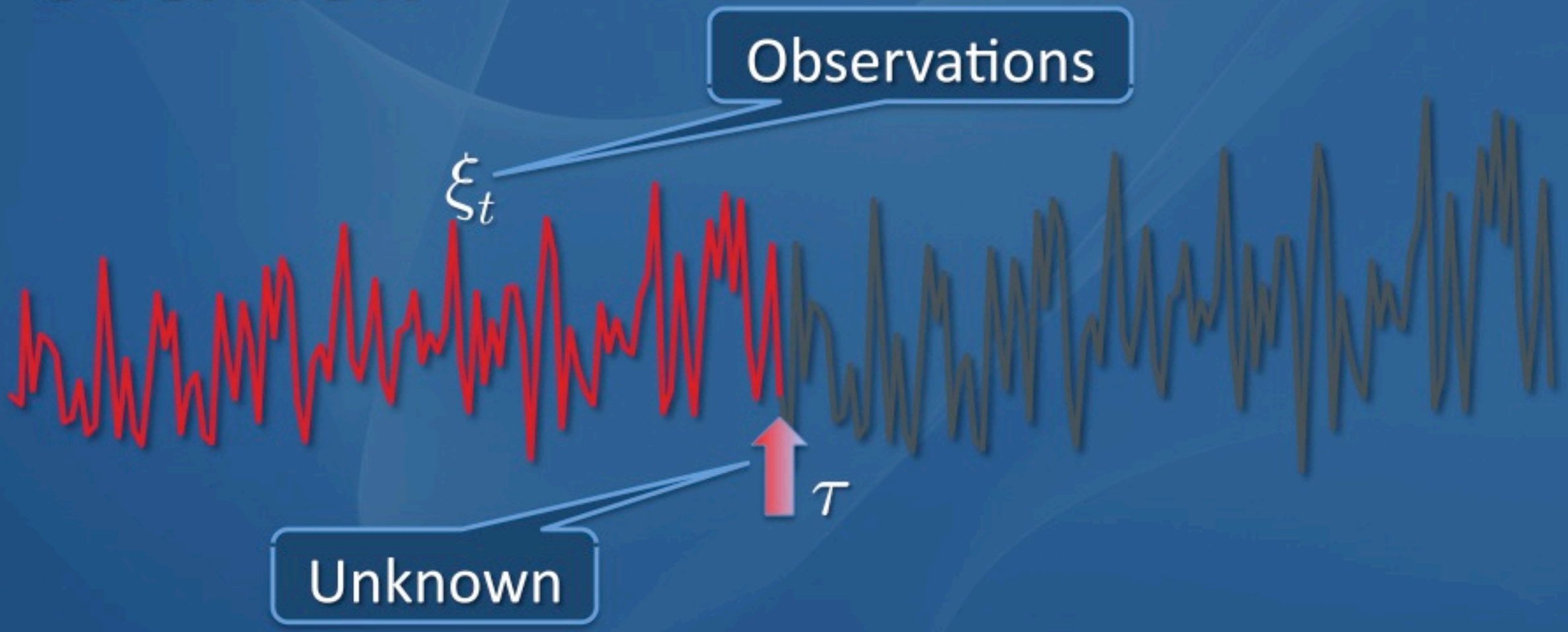
Sequential rate change detection in Poisson processes

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Outline

- Overview of sequential changepoint detection
- CUSUM test and Lorden's criterion
- The homogeneous Poisson disorder problem
  - CUSUM average run length
  - CUSUM optimality
- Non-homogeneous Poisson disorder problem and Epidemic surveillance

Overview



Using $\{\xi_t\}$ detect τ as soon as possible

Quality control
Systems monitoring
Remote sensing and GIS
Smart cameras – Human computer interaction
Image processing
⋮

Optical communications
Changepoint models for hazard functions
Occurrence of industrial accidents
Epidemic detection
Monitoring of link failures in computer networks
⋮



P_∞: nominal measure

P_0 : alternative measure

P_τ : measure induced by the change

E_τ [.]: corresponding expectation

Both P_0 and P_∞ are assumed known

We are interested in **sequential schemes**

With every new data point ξ_t we decide whether

- Stop and raise an alarm
- Continue sampling

For the decision at time t we use all the available information up to time t

$$\mathcal{F}_t = \sigma\{\, \xi_s : 0 \leq s \leq t \,\}$$

Sequential test $\Leftrightarrow$ stopping time T adapted to the filtration $\{\mathcal{F}_t\}$

Detection delay

what is τ ?

$$\mathcal{J}(T) = \mathsf{E}_\tau[T - \tau | T > \tau]$$

False alarm

Average period between false alarms:

$$\mathsf{E}_\infty[T] \geq \gamma$$

False alarm probability:

$$\mathsf{P}(T \leq \tau) \leq \alpha$$

$$\min_T \mathcal{J}(T)$$

Changepoint mechanism

There is a mechanism that **decides when** to impose the change. This decision can be

- **Independent** from the observations $\{\xi_t\}$
- **Depend** on the observations $\{\xi_t\}$

If independent, then τ appears as random variable with prior

$$\mathrm{P}(\tau = t) = \pi_t$$

If $\{\pi_t\}$ known then **Bayesian formulation**

Bayesian Formulation

Zero modified geometric

$$\pi_t = (1-\varpi)p(1-p)^{t-1}, t \geq 1; \quad \pi_0 = \varpi.$$

$$\min_T \mathsf{E}[T-\tau \,|\, T>\tau]; \text{ s.t. } \mathsf{P}(T \leq \tau) \leq \alpha$$

$$S_t = (S_{t-1}+1)\frac{f_0(\xi_t)}{(1-p)f_\infty(\xi_t)}, \quad S_0 = r$$

$$T_{\mathrm{SR}} = \inf\{t: \ S_t \geq \nu\}$$

I.i.d.: Shiryaev (1963,1978), Poor (1998). **BM**: Shiryaev (1961). **Poisson**: Peskir and Shiryaev (2002), Dayanik (2005,2006,2008)

The mechanism decides without consulting the observations but $\mathrm{P}(\tau = t) = \pi_t$ is **unknown**

$$\mathcal{J}_{\mathrm{P}}(T) = \sup_{\{\pi_t\}} \mathsf{E}_\tau[T - \tau | T > \tau]$$

$$= \sup_{t \geq 0} \mathsf{E}_t[T - t | T > t]$$

Pollak's (1985) criterion

$$\inf_T \mathcal{J}_{\mathrm{P}}(T) = \inf_T \sup_{t \geq 0} \mathsf{E}_t[T - t | T > t]$$

$$\text{s.t.} \quad \mathsf{E}_\infty[T] \geq \gamma$$

Pollak (1985) proposed the following stopping time known as Shiryaev-Roberts-Pollak

$$S_t = (S_{t-1} + 1)\frac{f_0(\xi_t)}{f_\infty(\xi_t)}, \quad S_0 \sim q(S)$$

$$T_{\mathrm{SRP}} = \inf\{t : S_t \geq \nu\}$$

It is an equalizer

$$\mathsf{E}_t[T_{\mathrm{SRP}} - t | T_{\mathrm{SRP}} > t] = \text{constant}$$

$$\mathcal{J}_{\mathrm{P}}(T_{\mathrm{SRP}}) - \inf_T \mathcal{J}_{\mathrm{P}}(T) = o(1), \quad \text{as } \gamma \to \infty$$

Strict Optimality ?

In 1997 appears a proof that the SRP test is optimum (Annals of Statistics)

Yajun Mei (2006), shows that the proof is problematic

The conjecture remained unanswered until last year (2010): Tartakovsky and Polunchenko, produced a counterexample. After 25 years we can finally say that

The SRP test is NOT optimum

CUSUM and Lorden's criterion

The changepoint mechanism decides to impose the change by consulting the observations $\{\xi_t\}$ and possibly additional information.

In this case τ becomes a stopping time adapted to a larger filtration than $\{\mathcal{F}_t\}$.

$$\mathcal{J}_{\mathrm{L}}(T) = \sup_{\tau} \mathsf{E}_\tau[T-\tau | T>\tau]$$

Lorden's (1971) criterion

$$= \sup_{t\geq 0} \operatorname{essup} \mathsf{E}_t[T-t | T>t, \mathcal{F}_t]$$

$$X_t = AX_{t-1} + BW_t$$

State space model

$$\xi_t = CX_t + Dw_t$$

Measurements

Possible cause for change is for example “large” oscillations

$$\tau = \inf\{t : \ \|X_t\| \geq c\}$$

$$A \rightarrow A'$$

A bomb set to explode at a specific time

$$\inf_T \mathcal{J}_{\mathrm{L}}(T) = \inf_T \sup_{t\geq 0} \operatorname{essup} \mathsf{E}_t[T-t|T>t,\mathcal{F}_t]$$

$$\mathsf{E}_\infty[T] \geq \gamma$$

The optimum scheme: CUSUM

$$u_t = \log\left(\frac{d\mathsf{P}_0}{d\mathsf{P}_\infty}(\mathcal{F}_t)\right)$$

$$m_t = \inf_{0\leq s\leq t} u_s$$

$$y_t = u_t - m_t \geq 0$$

$$T_{\mathrm{C}} = \inf\{t: \ y_t \geq \nu\}$$

I.i.d.: Moustakides (1986)
Ritov (1990), Poor (1998)

BM: Shiryaev (1996),
Beibel (1996)

Ito: Moustakides (2004)

ν

ν

ν

u_t

m_t

T_C

ML estimate of τ

Homogeneous Poisson disorder

Let $\{\mathcal{N}_t\}$ denote a homogeneous Poisson process with rate λ satisfying

$$\lambda = \begin{cases} \lambda_\infty & \text{for } t \le \tau \\ \lambda_0 & \text{for } t > \tau. \end{cases}$$

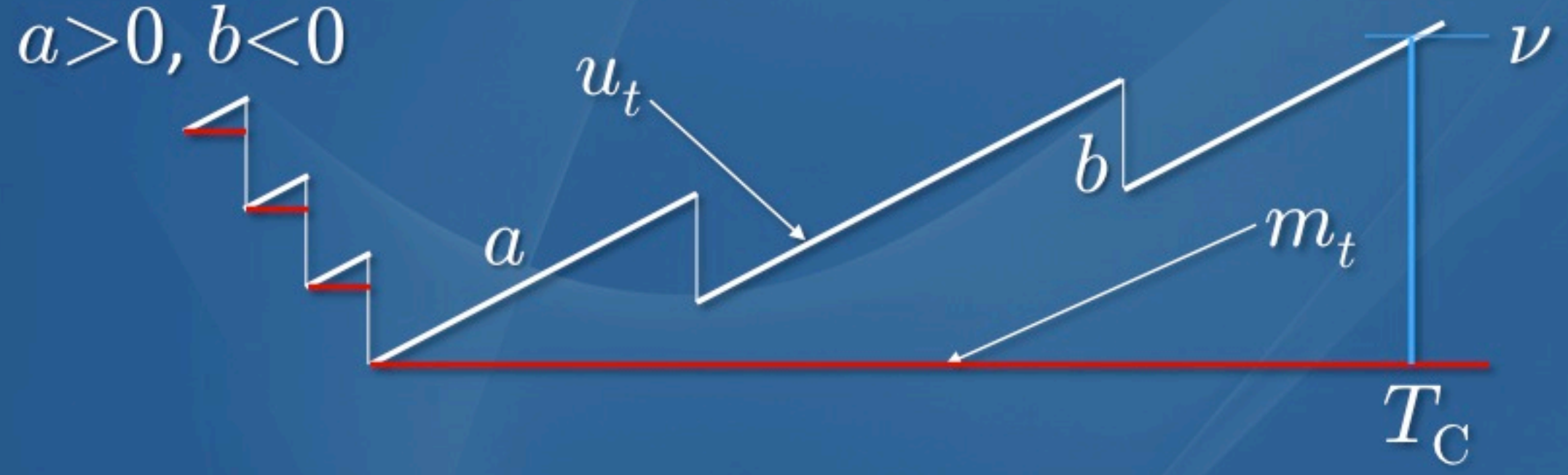
$$u_t = (\lambda_\infty - \lambda_0)t + \log\frac{\lambda_0}{\lambda_\infty}\mathcal{N}_t$$

$$u_t = a\,t + b\,\mathcal{N}_t$$

a,b opposite signs

Show optimality of CUSUM

First step compute ARL: $\mathrm{E}_0[T_\mathrm{C}]$ and $\mathrm{E}_\infty[T_\mathrm{C}]$



We are looking for $f(y)=\mathsf{E}[T_{\mathrm{C}}\,|\,y_0=y]$

$$\mathsf{E}[f(y_{T_{\mathrm{C}}})]-f(y_0)=$$

$$\mathsf{E}\left[\int_0^{T_{\mathrm{C}}}\{\underbrace{af'(y_{t-})+\lambda[f(y_{t-}+b)-f(y_{t-})]}_{=-1}\}dt\right]$$

$$af'(y) + \lambda[f(y+b) - f(y)] = -1$$

$$f(\nu) = 0; \;\; f(y) = f(0) \;\; y \leq 0$$

$$f(y) = \frac{1}{\lambda}\left\{\sum_{n=0}^{[\frac{\nu}{|b|}]} \phi_n(\nu - n|b|) - \sum_{n=0}^{[\frac{y}{|b|}]} \phi_n(y - n|b|)\right\}$$

$$\phi_n(y) = e^{\frac{\lambda y}{a}}\left(\sum_{k=0}^{n} \frac{\left(\frac{-\lambda y}{a}\right)^k}{k!}\right) - 1;$$

Optimality of CUSUM

$$\mathcal{J}_{\mathrm{L}}(T) = \sup_{t\geq 0} \operatorname{essup} \mathsf{E}_t[T - t | T > t, \mathcal{F}_t]$$

$$\mathsf{E}_\infty[T] \geq \gamma$$

$$\mathsf{E}_\infty[T] \geq \mathsf{E}_\infty[T_{\mathrm{C}}] = \gamma \quad \Rightarrow \mathcal{J}_{\mathrm{L}}(T) \geq \mathcal{J}_{\mathrm{L}}(T_{\mathrm{C}})$$

$$h(y) = \mathsf{E}_\infty[T_{\mathrm{C}} | y_0 = y]$$

$$g(y) = \mathsf{E}_0[T_{\mathrm{C}} | y_0 = y]$$

$$\operatorname{essup} \mathsf{E}_t[T_{\mathrm{C}} - t | T_{\mathrm{C}} > t, \mathcal{F}_t] = g(0)$$

Equalizer

$$\mathcal{J}_{\mathrm{L}}(T_{\mathrm{C}}) = g(0) \qquad \mathsf{E}_\infty[T_{\mathrm{C}}] = h(0)$$

$$\mathsf{E}_\infty[T] \geq h(0) \Rightarrow \mathcal{J}_{\mathrm{L}}(T) \geq g(0)$$

Lemma: $$\mathcal{J}_{\mathrm{L}}(T) \geq \frac{\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt]}{\mathsf{E}_\infty[e^{y_T}]}$$

$$\mathsf{E}_\infty[T] \geq h(0) \Rightarrow \frac{\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt]}{\mathsf{E}_\infty[e^{y_T}]} \geq g(0)$$

$$\mathsf{E}_\infty[\int_0^T e^{y_t}\,dt] = g(0)\mathsf{E}_\infty[e^{y_T}] + \underbrace{\mathsf{E}_\infty[T] - h(0)}_{\geq 0}$$

$$+ \underbrace{\mathsf{E}_\infty[h(y_T) - e^{y_T} g(y_T)]}_{\geq 0}$$

Lemma: $$h(y) - e^y g(y) \geq 0, \quad \forall\ y \geq 0$$

Nonhomogeneous Poisson

Let $\{\mathcal{N}_t\}$ denote a nonhomogeneous Poisson process with rate λ_t that satisfies

$$\lambda_t = \begin{cases} \omega_t & \text{for } t \leq \tau \\ \rho\,\omega_t & \text{for } t > \tau, \end{cases}$$

ω_t is adapted to our observations (that can include more information than $\{\mathcal{N}_t\}$) and ρ is a known constant.

CUSUM test:

$$u_t = (1-\rho)\int_0^t \omega_s\, ds + (\log\rho)\mathcal{N}_t$$

$$m_t = \inf_{0\le s\le t} u_s$$

$$y_t = u_t - m_t$$

$$T_{\mathrm{C}} = \inf\{t:\ y_t \ge \nu\}$$

Is it optimum? **YES !** but in a Lorden-like sense

$$\mathcal{J}_{\mathrm{L}}(T) = \sup_{t\ge 0} \operatorname{essup} \mathsf{E}_t[\mathcal{N}_T - \mathcal{N}_t | T > t, \mathcal{F}_t]$$

$$\mathsf{E}_\infty[\mathcal{N}_T] \ge \gamma$$

Epidemic Detection

Detect the onset of an epidemic: when the **incidence rate** of a particular disease increases significantly above some standard level

ω_t : describes the nominal incidence rate, which can be a function of several observable quantities as population, pollution measurements etc.

$\rho > 1$: smallest increase in incidence rate that should be considered … alarming.

Detect rapidly **in number of incidences**

Seminars

2009-10 seminars 2008-09 seminars 2007-08 seminars 2006-07 seminars 2005-06 seminars

Seminars 2010-2011

Upcoming

WEDNESDAY January 12, South Hall 5607F, 3:30 PM, Refreshments served at 3:15 PM

Dr. Mike Ludkovski and Dr. Jarad Niemi (UCSB Statistics and Applied Probability)

Title: Optimal Dynamic Policies for Influenza Management

Abstract: Management policies for influenza outbreaks balance the expected morbidity and mortality costs versus the cost of intervention policies. We present a methodology for dynamic determination of optimal policies in stochastic compartmental models with parameter uncertainty. Our approach is simulation-based and searches the full set of sequential control strategies. For each time point, it generates a policy map describing the optimal intervention to implement as a function of outbreak state and Bayesian parameter posteriors. As a running example, we study a fully observed stochastic SIR model with isolation and vaccination as two possible interventions. Numerical simulations based on a classic influenza outbreak are used to explore the impact of various cost structures on management policies. We also discuss more complex inference schemes based on incomplete observations.

This talk is geared towards graduate students and will combine both statistical and probabilistic techniques.

Paper URL: http://www.bepress.com/scid/vol2/iss1/art5

WEDNESDAY January 19, South Hall 5607F, 3:30 PM, Refreshments served at 3:15 PM

Prof. George Moustakides (University of Patras, Greece)

Title: Sequential rate change detection in Poisson processes

Abstract: We consider the Cumulative Sum (CUSUM) test as a possible candidate for sequential detection of an abrupt change in the rate of a homogeneous Poisson process. We first derive a closed form expression for the average run length of the CUSUM stopping time, which we use to prove optimality of the CUSUM test in the sense of Lorden. Specifically, we demonstrate that the CUSUM stopping time minimizes the maximal possible conditional detection delay under the constraint that the average period between false alarms is no less than a prescribed value. We then consider a special category of rate changes in non-homogeneous Poisson processes and we show that the optimality of CUSUM extends to this more general class, provided that we suitably modify our initial performance measure. We conclude our presentation by discussing the applicability of the non-homegeneous Poisson detection problem to Epidemic surveillance.

WEDNESDAY February 2, South Hall 5607F, 3:30 PM, Refreshments served at 3:15 PM

Prof. Dongseok Choi (Oregon Health and Science University)

Further information to be announced

WEDNESDAY February 9, South Hall 5607F, 3:30 PM, Refreshments served at 3:15 PM

Prof. Olympia Hadjiliadis (CUNY)

Further information to be announced

Past

WEDNESDAY October 6, South Hall 5607F, 3:30 PM, Refreshments served at 3:15 PM

Prof. Sreenivas Jammalamadaka (UCSB Statistics and Applied Probability)

Title: Gaps between Observations – What can one learn from them?

Abstract: This talk will provide an overview of some of the main ideas in the theory of spacings, i.e. the gaps between successive observations. After reviewing some basic properties of spacings, their use in testing statistical hypotheses and in estimating