Detecting changes in the rate

of a

Poisson process

George V. Moustakides

Outline

- Overview of the change detection problem
- CUSUM test and Lorden's criterion
- The Poisson disorder problem
  - CUSUM average run length for Poisson processes
  - CUSUM optimality in the sense of Lorden

Change detection - Overview

Available sequentially an observation process $\{\xi_t\}$ with the following statistics:

$$\xi_t \sim \mathbb{P}_\infty \quad \text{for } 0 \leqslant t \leqslant \tau$$
$$\sim \mathbb{P}_0 \quad \text{for } \tau < t$$

Detect change **as soon as possible**

- Change time τ :
  - Random with known prior (Bayesian)
  - Deterministic but unknown (Non-Bayesian)
- Known statistics $\mathbb{P}_\infty$, $\mathbb{P}_0$.

We are interested in **sequential schemes**.

With every new observation the test must decide

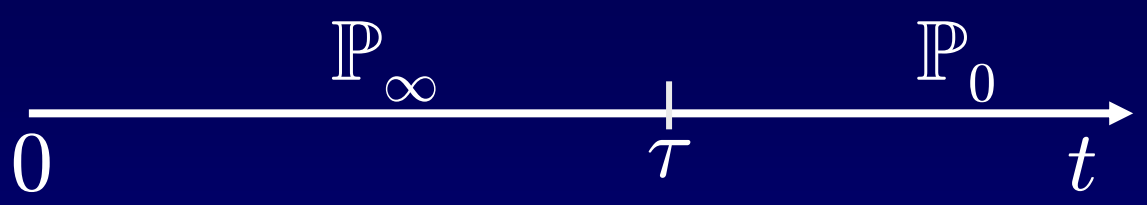
- **Stop** and issue an alarm
- Continue sampling

Decision at time t uses available information

$$\mathcal{F}_t = \sigma\{\xi_s : 0 \leqslant s \leqslant t\}.$$

up to time t.

Sequential test → stopping time T adapted to the filtration $\{\mathcal{F}_t\}$.



$\mathbb{P}_\tau$: the probability measure induced, when change takes place at time τ

$\mathbb{E}_\tau[.]$: the corresponding expectation

$\mathbb{P}_\infty$: all data under nominal regime

$\mathbb{P}_0$: all data under alternative regime

Parameters to be considered

- The detection delay T - τ
- Frequency of false alarms

Bayesian approach (Shiryayev 1978)

Change time τ random with exponential prior.

$$J(T) = c\,\mathbb{E}[\ (T - \tau)^{+}\ ] + \mathbb{P}[\ T < \tau\ ]$$

Optimization problem: $\inf_T J(T)$

$$\pi_t = \mathbb{P}[\ \tau \leqslant t \mid \mathcal{F}_t]; \qquad T_{\mathrm{S}} = \inf_t \{\ t\colon \pi_t \geqslant \nu\ \}$$

- Discrete time: i.i.d. observations (Shiryayev 1978, Poor 1998)
- Continuous time: Brownian Motion (Shiryayev 1978, Beibel 2000, Karatzas 2003)

Non-Bayesian setup (Pollak 1985)

The change time τ is deterministic & unknown.

$$J(T) = \sup_{\tau} \mathbb{E}_{\tau}[\,(T - \tau) \,|\, T > \tau\,]$$

Optimization problem: $\inf_T J(T)$

subject to: $\mathbb{E}_{\infty}[\,T\,] \geqslant \gamma$

Discrete time: i.i.d. detect change in the pdf from $f_{\infty}(\xi)$ to $f_0(\xi)$. Roberts (1966) proposed

$$S_t = (S_{t\text{-}1} + 1)\,\frac{f_0(\xi_t)}{f_{\infty}(\xi_t)}$$

$$T_{\mathrm{SRP}} = \inf_t \{\, t\colon S_t \geqslant \nu \,\}$$ (Mei 2006)

CUSUM test and Lorden's criterion

Discrete time, i.i.d. observations.
Pdf before and after the change: $f_\infty(\xi_n)$, $f_0(\xi_n)$

Since change time τ is unknown

$$\sup_{0\leqslant\tau\leqslant t} \sum_{n=\tau+1}^{t} \log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right) \geqslant \nu$$

$$\sum_{n=1}^{t} \log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right) - \inf_{0\leqslant\tau\leqslant t} \sum_{n=1}^{\tau} \log\left(\frac{f_0(\xi_n)}{f_\infty(\xi_n)}\right) \geqslant \nu$$

$$u_t \quad - \quad m_t \quad \geqslant \nu$$

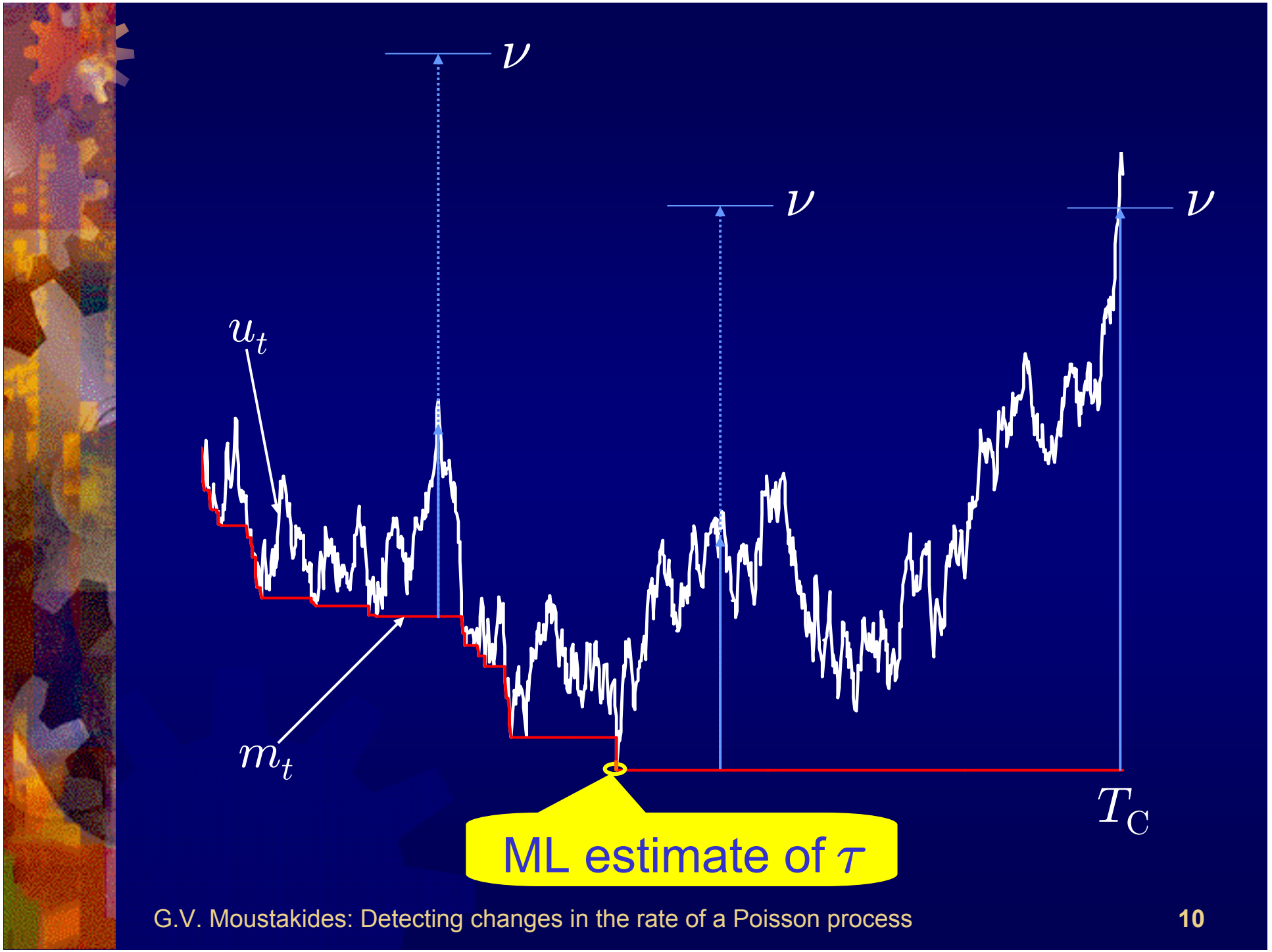
$$u_t = \log\Big(\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty}(\mathcal{F}_t) \Big)$$

$$m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

CUSUM process: $y_t = u_t - m_t \geqslant 0$

The CUSUM stopping time (Page 1954):

$$T_{\mathrm{C}} = \inf_t \{\, t\colon y_t \geqslant \nu \,\}$$


ν
ν
ν
u_t
m_t
T_C
ML estimate of τ

Non-Bayesian setup (Lorden 1971).

Change time τ is deterministic and unknown.

$$J(T) = \sup_{\tau} \operatorname{essup} \mathbb{E}_{\tau}[\ (T - \tau)^{+} \mid \mathcal{F}_{\tau}\ ]$$

Optimization problem: $\inf_{T} J(T)$

subject to: $\mathbb{E}_{\infty}[\,T\,] \geqslant \gamma$

- Discrete time: i.i.d. observations (Moustakides 1986, Ritov 1990, Poor 1998)
- Continuous time: BM (Shiryayev 1996, Beibel 1996); Ito processes (Moustakides 2004)

The Poisson disorder problem

Let $\{\mathcal{N}_t\}$ homogeneous Poisson, with rate λ satisfying:

$$\lambda = \begin{cases} \lambda_\infty\,, & 0 \leqslant t \leqslant \tau \\ \lambda_0\,, & \tau < t \end{cases}$$

Bayesian Approach

- Linear delay: Galchuk & Rozovski 1971, Davis 1976, Peskir & Shiryayev 2002.
- Exponential delay: Bayraktar & Dayanik 2003, B & D & Karatzas 2004 and 2005 (adaptive)

CUSUM & average run length

$$u_t = (\lambda_\infty - \lambda_0)t + \log(\lambda_0/\lambda_\infty)\mathcal{N}_t$$

$$m_t = \inf_{0 \leqslant s \leqslant t} u_s$$

$$y_t = u_t - m_t$$

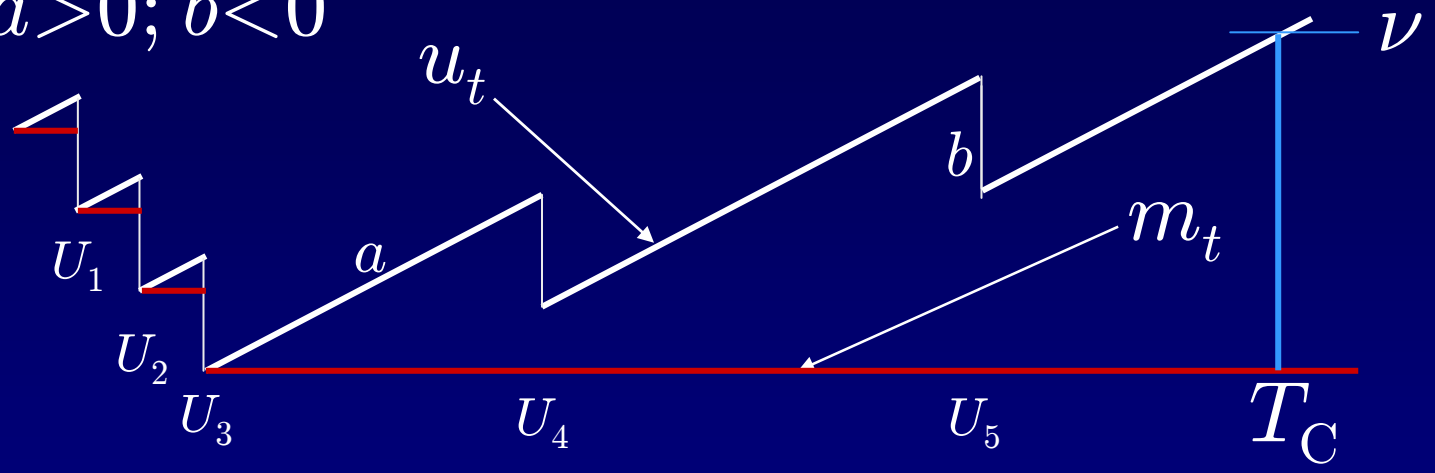
$$T_{\mathrm{C}} = \inf_t \{\ t\colon y_t \geqslant \nu\ \}$$

We are interested in computing $\mathbb{E}[\ T_{\mathrm{C}}\ ]$ when $\mathcal{N}_t$ is Poisson with rate λ.

Existing formula (Taylor 1975) for:

$u_t = at + b\mathcal{W}_t$; $\mathcal{W}_t$ is standard Wiener

$a>0;\ b<0$



Find $f(y)$ such that $f(y_0)=\mathbb{E}[T_C]$.

Study the paths of $f(y_t)$

$$f(y_t) - f(y_0) = \int_0^t f'(y_{s-})a\,ds + \int_0^t [f(y_{s-}+b) - f(y_{s-})]\,d\mathcal{N}_s$$

$$f(y)=f(0);\ y\leqslant 0$$

$$f(y_{T_C}) - f(y_0) = \int_0^{T_C} f'(y_{t-})a\,dt + \int_0^{T_C} \left[f\left(y_{t-}+b\right) - f(y_{t-})\right]\, d\mathcal{N}_t$$

$$\mathbb{E}[f(y_{T_C})] - f(y_0) =$$

$$\mathbb{E}\left[\int_0^{T_C}\left\{f'(y_{t-})a + \lambda[f\left(y_{t-}+b\right) - f(y_{t-})]\right\}dt\right]$$

$$af'(y) + \lambda[f\left(y+b\right) - f(y)] = -1;\ \ y \in [0, \nu)$$

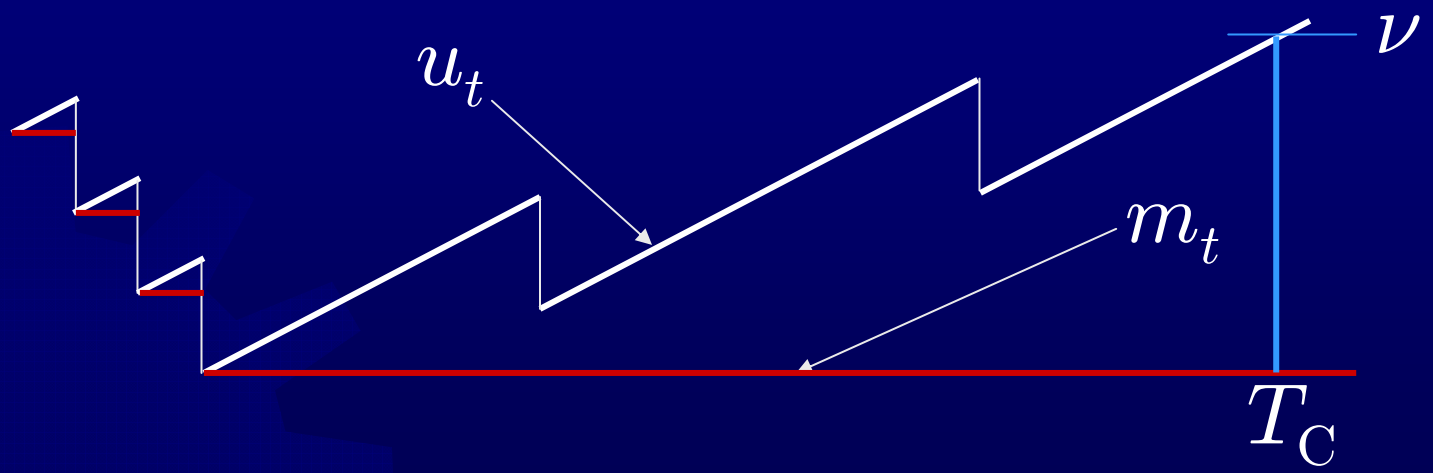
$$f(y_0) - \mathbb{E}[f(y_{T_C})] = \mathbb{E}[T_C]$$

We end up with a DDE and the following boundary conditions:

$$a f'(y) + \lambda[f(y+b) - f(y)] = -1;\ y \in [0,\nu)$$

$$f(y) = f(0) \text{ for } y \leqslant 0; \qquad f(\nu) = 0$$

$$f(y_0) = \mathbb{E}[T_{\mathrm{C}}]$$



Existing formula (Taylor 1975) for:

$u_t = at + b\mathcal{W}_t$; $\mathcal{W}_t$ standard Wiener

$$\frac{b^2}{2} f''(y) + a f'(y) = -1;\ y \in [0,\nu]$$

$$f'(0) = 0; \qquad f(\nu) = 0$$

$$\mathbb{E}[T_C | y_0 = y] = f(y) =$$

$$\frac{1}{a}\left\{\frac{b^2}{2a}\left(1 - e^{-\frac{2a}{b^2}\nu}\right) - \nu\right\}$$

$$-\frac{1}{a}\left\{\frac{b^2}{2a}\left(1 - e^{-\frac{2a}{b^2}y}\right) - y\right\}$$

$$af'(y) + \lambda[f(y+b) - f(y)] = -1;\ y\in[0,\nu)$$

$$f(y) = f(0) \text{ for } y \leqslant 0; \qquad f(\nu)=0$$

Because $b<0$ it is a forward DDE

$$\mathbb{E}[T_C|y_0 = y] = f(y) =$$

$$\frac{1}{\lambda}\sum_{n=0}^{\left[\frac{\nu}{|b|}\right]}\left\{e^{\frac{\lambda(\nu-n|b|)}{a}}\left(\sum_{k=0}^{n}\frac{\left(\frac{-\lambda(\nu-n|b|)}{a}\right)^k}{k!}\right) - 1\right\}$$

$$-\frac{1}{\lambda}\sum_{n=0}^{\left[\frac{y}{|b|}\right]}\left\{e^{\frac{\lambda(y-n|b|)}{a}}\left(\sum_{k=0}^{n}\frac{\left(\frac{-\lambda(y-n|b|)}{a}\right)^k}{k!}\right) - 1\right\}$$

$a<0; \quad b>0$

$af'(y) + \lambda[f(y+b) - f(y)] = -1; \; y\in[0,\nu)$

$f'(0)=0;$ $\qquad\qquad$ $f(y)=0$ for $y \geqslant \nu$

Backward DDE

$$\mathbb{E}[T_C|y_0 = 0] = f(0) =$$

$$\frac{1}{\lambda}\sum_{n=0}^{[\frac{\nu}{b}]}\left\{1 - e^{\frac{\lambda(\nu-nb)}{|a|}}\sum_{k=0}^{n}\frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^k}{k!}\right\}$$

$$+\frac{p}{\lambda}\sum_{n=0}^{[\frac{\nu}{b}]} e^{\frac{\lambda(\nu-nb)}{|a|}}\frac{\left(-\frac{\lambda(\nu-nb)}{|a|}\right)^n}{n!}$$

where p is defined as

$$p = \frac{A}{A - B}$$

with

$$A = \sum_{n=0}^{\left[\frac{\nu}{b}\right]} e^{\frac{\lambda(\nu - nb)}{|a|}} \frac{\left(-\frac{\lambda(\nu - nb)}{|a|}\right)^n}{n!}$$

$$B = \sum_{n=1}^{\left[\frac{\nu}{b}\right]} e^{\frac{\lambda(\nu - nb)}{|a|}} \frac{\left(-\frac{\lambda(\nu - nb)}{|a|}\right)^{n-1}}{(n-1)!}$$

Average over 10000 repetitions:

$\lambda_\infty = 2, \;\; \lambda_0 = 1, \; (a{=}1, \; b{=}\text{-}\log 2), \; \nu = 5.5$

	Formula	Simulation
$\mathbb{E}_0[\, T_C \,]$	15.3832	15.3605
$\mathbb{E}_\infty[\, T_C \,]$	779.9669	771.1219

$\lambda_\infty = 1, \;\; \lambda_0 = 2, \; (a{=}\text{-}1, \; b{=}\log 2), \; \nu = 5.5$

	Formula	Simulation
$\mathbb{E}_0[\, T_C \,]$	12.2885	12.2673
$\mathbb{E}_\infty[\, T_C \,]$	981.9811	986.7159

Optimality of CUSUM

$$J(T) = \sup_{\tau} \operatorname{essup} \mathbb{E}_{\tau}[\ (T - \tau)^{+} \mid \mathcal{F}_{\tau}\ ]$$

$\inf_T J(T)$; subject to: $\mathbb{E}_{\infty}[\ T\ ] \geqslant \gamma$

If T is such that

$$\mathbb{E}_{\infty}[\ T\ ] \geqslant \mathbb{E}_{\infty}[\ T_{\mathrm{C}}\ ] = \gamma$$

then

$$J(T) \geqslant J(T_{\mathrm{C}})$$

$$h(y) = \mathbb{E}_{\infty}[\, T_{\mathrm{C}} \,|\, y_0{=}y]$$

$$g(y) = \mathbb{E}_{0}[\, T_{\mathrm{C}} \,|\, y_0{=}y]$$

$$\operatorname{essup} \mathbb{E}_{\tau}[\,(T_{\mathrm{C}} - \tau)^{+} \,|\, \mathcal{F}_{\tau}] = \sup_{y} g(y) = g(0)$$

T_{C} is an **equilizer rule** therefore

$$J(T_{\mathrm{C}}) = g(0)$$

For the false alarm we have

$$\mathbb{E}_{\infty}[\, T_{\mathrm{C}} \,] = h(0)$$

We would like to show:

If $\mathbb{E}_\infty[T] \geqslant \mathbb{E}_\infty[T_{\mathrm{C}}]$

then $J(T) \geqslant J(T_{\mathrm{C}})$

Lemma

$$J(T) \geq \frac{\mathbb{E}_\infty\left[\int_0^T e^{y_s} ds\right]}{\mathbb{E}_\infty\left[e^{y_T}\right]} = \mathcal{J}(T)$$

Sufficient:

If $\mathbb{E}_\infty[T] \geqslant h(0)$

then $\mathcal{J}(T) \geqslant g(0)$

$$\mathcal{J}(T) = \frac{\mathbb{E}_\infty \left[\int_0^T e^{y_s} ds \right]}{\mathbb{E}_\infty \left[e^{y_T} \right]} \geq g(0)$$

$$\mathbb{E}_\infty \left[\int_0^T e^{y_s} ds - g(0) e^{y_T} \right]$$

$$-\{\mathbb{E}_\infty[T] - h(0)\} \geq 0$$

$$\mathbb{E}_\infty \left[\int_0^T (e^{y_s} - 1) ds - g(0) e^{y_T} + h(0) \right] \geq 0$$

We will show that this is true for **any** T

Consider the function $f(y)$ defined as follows

$$f(y)=e^{y}\left[g(0)\text{ - }g(y)\right]\text{ - }\left[h(0)\text{ - }h(y)\right]$$

then

$$\mathbb{E}_{\infty}[f(y_T)]=\mathbb{E}_{\infty}\left[\int_0^T(e^{y_s}-1)1_{\{y_s<\nu\}}ds\right]$$

$$\mathbb{E}_{\infty}\left[\int_0^T(e^{y_s}-1)ds-g(0)e^{y_T}+h(0)\right]\overset{?}{\geqslant}0$$

$$=\mathbb{E}_{\infty}\left[\int_0^T(e^{y_s}-1)1_{\{y_s\geq\nu\}}ds\right]$$

$$+\mathbb{E}_{\infty}\left[h(y_T)-e^{y_T}g(y_T)\right]\geqslant 0$$



Conclusion

- We considered the Poisson disorder problem of detecting changes in the rate of a homogeneous Poisson process, in the sense of Lorden.
- We obtained closed form expressions for the average run length of the CUSUM stopping time.
- We used these formulas to prove optimality of the CUSUM test in the sense of Lorden.

endar of events

February 6-12, 2006

Previous calendar Next calendar

[F] Admission charged, [G] Not open to general public.
All other events are open to members of the University community and the general public free of charge. Any speaker not otherwise identified is a member of the faculty, staff or student body of Princeton University. • Contact Calendar editor • Submissions for future calendars may be made online by completing the calendar submission form. • For copy deadlines, please refer to the PWB deadline schedule.

Mon | Tue | Wed | Thu | Fri | Sat | Sun | Weekly Exhibits Etc Top

Monday, February 6

Arts

7:30 p.m. Center for Human Values film. Luis Buñuel: "The Discreet Charm of the Bourgeoisie." Josiah Ober, speaker. Theater, Rockefeller and Mathey College.

Lectures

4:30 p.m. Near Eastern studies Islamic seminar. "Grandfathers Rediscovered: Sufism in Contemporary Kabylia." Judith Scheele, University of Oxford. 102 Jones.
6 p.m. School of Architecture lecture. "Playgrounds." Luis Mansilla and Emilio Tuñón, Mansilla + Tuñón Arquitectos, Madrid. Betts Auditorium, School of Architecture.

Notices

[G] 4:30 p.m. Faculty meeting. 101 McCormick.
7 p.m. University Store book reading and signing. Matthew Stewart, author of "The Courtier and the Heretic." University Store.

Mon | Tue | Wed | Thu | Fri | Sat | Sun | Weekly Exhibits Etc Top

Tuesday, February 7

Arts

[F] 8 p.m. McCarter Theatre performance. King's Singers. Matthews Theatre.

Lectures

Noon. Population research seminar. "Discrimination in Low Wage Labor Markets." Devah Pager. 300 Wallace.
4 p.m. Chemistry seminar. "Charge Transport in Molecular Monolayers, Multilayers and Thin Films." Cherie Kagan, IBM T.J. Watson Research Center. DuPont Seminar Room, 324 Frick.
4:15 p.m. Astrophysical sciences astronomy colloquium. "The Triple Nucleus and Supermassive Black Hole of M31." Ralf Bender, Max-Planck-Institut fuer Extraterrestrische Physik, Germany. Auditorium, Peyton. Social gathering at 5:15 p.m., main hallway.
4:30 p.m. Mathematics algebraic geometry seminar. Grigory Mikhalkin, University of Toronto. 322 Fine.
4:30 p.m. Operations research and financial engineering seminar. "Detecting Changes in the Rate of a Poisson Process." George Moustakides, University of Thessaly, Greece. E219 Engineering Quadrangle.
4:30 p.m. Woodrow Wilson School/graduate career services lecture. "Business and Finance at

SEMINARS
Updated: 1-18-2006

JANUARY

Special Seminar

Topic: Symmetric tensors and enumeration of Hamiltonian cycles in graphs
Presenter: **Peter Zograf**, Steklov Mathematical Institute
Date: Friday, January 20, 2006, Time: 2:15 p.m., Location: Fine Hall 314
Abstract: A Hamiltonian cycle is a closed path that traverses each vertex of a graph exactly once. The problem of finding a Hamiltonian cycle in a graph is a well known NP-hard problem, a discrete relative of the famous Traveling Salesman Problem. Even to decide whether a graph contains a Hamiltonian cycle (without actually finding it) is computationally NP-hard.
This talk will explain how one can enumerate Hamiltonian cycles in a graph by contracting certain symmetric tensors along the edges of the graph. The method is rather general and applies to other enumeration problems in graphs, like edge colorings, perfect matchings, etc.

Analysis Seminar *** Please note special date

Topic: The structure of entropy solutions of nonlinear scalar conservation laws
Presenter: **Felix Otto**, Institute for Applied Mathematics, University of Bonn
Date: Wednesday, January 25, 2006, Time: 4:00 p.m., Location: Fine Hall 214

Geometric Analysis Seminar

Topic: Holomorphic vector fields and deformation rigidity
Presenter: **Ngaiming Mok**, The University of Hong Kong
Date: Friday, January 27, 2006, Time: 3:00 p.m., Location: Fine Hall 314

FEBRUARY

Algebraic Geometry Seminar

Topic: TBA
Presenter: **G. Mikhalkin,** University of Toronto
Date: Tuesday, February 7, 2006, Time: 4:30 p.m., Location: Fine Hall 322

Operation Research and Financial Engineering Seminar

Topic: Detecting changes in the rate of a Poisson process
Presenter: **George Moustakides**, University of Thessaly
Date: Tuesday, February 7, 2006, Time: 4:30 p.m., Location: Room E-219, Engineering Quad
Abstract: We consider the Cumulative Sum (CUSUM) test as a possible candidate to sequentially detect a change in the rate of a homogeneous Poisson process. We first derive a closed form

expression for the average run length of the CUSUM stopping time, which we then use to prove optimality of the CUSUM test in the sense of Lorden. Specifically, we demonstrate that the CUSUM stopping time minimizes the maximal possible conditional detection delay under the constraint, that the average period between false alarms is no less than a prescribed value.

Geometric Analysis Seminar

Topic: TBA
Presenter: **Christina Sormani**, City University of New York
Date: Friday, February 10, 2006, Time: 3:00 p.m., Location: Fine Hall 314

PACM Seminar

Topic: Fault-Tolerant Quantum Computation
Presenter: **Barbara Terhal**, IBM
Date: Monday, February 13, 2006, Time: 4:00 p.m., Location: Fine Hall 214
Abstract: I will review the theory of fault-tolerant quantum computation and the use of quantum error-correcting codes in future quantum computers. I will discuss the most recent developments in this area.

Algebraic Geometry Seminar

Topic: TBA
Presenter: **V. Alexeev**, University of Georgia
Date: Tuesday, February 14, 2006, Time: 4:30 p.m., Location: Fine Hall 322

Operation Research and Financial Engineering Seminar

Topic: TBA
Presenter: **Michael Overton**, New York University
Date: Tuesday, February 14, 2006, Time: 4:30 p.m., Location: Room E-219, Engineering Quad

Geometric Analysis Seminar

Topic: TBA
Presenter: **Natasa Sesum** , Columbia University
Date: Friday, February 17, 2006, Time: 3:00 p.m., Location: Fine Hall 314

PACM Seminar

Topic: Math Problems from the Far Side of Quantum Information
Presenter: **Christopher A. Fuchs**, Bell Labs, Lucent Technologies
Date: Monday, February 20, 2006, Time: 4:00 p.m., Location: Fine Hall 214
Abstract: The field of Quantum Information has recently rightly attracted great interest for the technological fruits it may bear. But there is a sect of its practitioners who think it stands a chance to bring us much more than that---namely, that its theoretical tools will give us a means for exploring what quantum mechanics is really all about and for settling some of the deepest problems in physics. The roots of this optimism come from a very old thought: that a quantum state has more to do with representing its user's information, than any inherent physical